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TWO-DIMENSIONAL THEORY OF ELASTICITY FOR FINITE DEFORMATIONS

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A general theory of plane stress, valid for large elastic deformations of isotropic materials, is developed using a general system of co-ordinates. No restriction is imposed upon the form of the strain-energy function in the formulation of the basic theory, which follows similar lines to the treatment by Adkins, Green & Shield (1953) of finite plane strain. The reduction of the equations to two-dimensional form subsequent to the assumption of plane stress enables the theory to be presented in complex variable notation.

A method of successive approximation is evolved, similar to that developed for problems in plane strain, which may be applied when exact solutions are not readily obtainable. The stress and displacement functions are expressed in terms of complex potential functions, and in the present paper the approximation process is terminated when the second-order terms have been obtained. The theory is formulated initially in terms of a complex co-ordinate system related to points in the deformed body, and the corresponding results for complex co-ordinates in the undeformed body are then obtained by a simple change of independent variable. Approximation methods are also applied to compressible materials in plane strain, and it is shown that the second-order terms for plane stress and plane strain can be expressed in similar forms. This leads to a general formulation of the second-order theory for two-dimensional problems, the results for plane stress or plane strain being derived by introducing the appropriate constants into the expressions thus obtained.

1. Introduction

The non-linearity of the differential equations which arise in formulating the mathematical theory of elasticity for large deformations has so far restricted the range of problems which have received satisfactory treatment to those in which marked simplifying features can be introduced. For example, in the problems of torsion, shear and flexure solved by Rivlin

(1948, 1949*a*, *b*,) for incompressible materials, and the inflation of a spherical shell examined by Green & Shield (1950), the restriction upon the form of deformation is such that, with the appropriate choice of co-ordinate system, the resulting equations can be solved quite generally for any form of strain-energy function. In other cases, as, for instance, in the analysis of the shear of a cylindrical annulus given by Rivlin (1949*b*), and in the generalizations of this problem examined by Adkins (1954), solutions have been obtained by assuming in addition a simplified form of strain-energy function, such as that postulated by Mooney (1940) for rubberlike materials.

It is evident that any general restriction upon the form of deformation is likely to produce some simplification in the form of the resulting equations. Thus if a problem can be reduced to two-dimensional form, some measure of simplicity will be achieved owing to the reduction of the number of dependent and independent variables which need to be considered, and into this category come the practically important problems of plane stress and plane strain.

The general theory of plane strain for large elastic deformations of isotropic materials has already been formulated by Adkins, Green & Shield (1953). In the present paper the corresponding theory is developed for plane stress. The undeformed body is assumed to consist of a thin plane uniform plate of isotropic elastic material which is stretched by forces in its plane so that it remains plane after deformation. When no forces act on the major surfaces of the plate, it is assumed, as in the classical theory of plane stress, that the principal stress component acting normally to the middle plane of the plate vanishes everywhere. The deformation and the stress resultants at any point are then expressed approximately as functions of position on the middle surface of the plate. Similar methods have been employed by Rivlin & Thomas (1951), and by Adkins & Rivlin (1952) in dealing with large deformations of thin sheets of incompressible materials, but in the problems there considered the resulting equations have been simplified by symmetry considerations.

In developing the general theory of plane stress, the stress resultants are expressed in terms of an Airy stress function ϕ chosen to satisfy the equations of equilibrium, and the work of §§3 and 4 is similar to that of Green & Zerna (1954) on the classical theory of thin plates. The resulting equations obtained in §§5 and 6 bear a formal resemblance to the corresponding equations for finite plane strain superposed on a uniform finite extension obtained by Adkins *et al.* (1953), but an additional unknown variable is introduced owing to the variation of thickness throughout the deformed plate.

For unsymmetrical problems, where exact solutions are not readily obtainable by orthodox methods of approach, second-order solutions, valid for a limited range of deformation, may be obtained by approximation methods. Some simple deformations of compressible materials have been investigated by such methods by Murnaghan (1937, 1951) and general formulations of the second-order theory have been given by Rivlin (1953) and by Green & Spratt (1954). Torsion problems have been similarly examined by a number of workers including Green & Shield (1951), and Green & Wilkes (1953). The method adopted in the present paper is similar to that employed by Adkins *et al.* (1953) for finite plane strain. It is assumed that the stress and displacement functions can be expressed as functions of a characteristic real parameter ϵ, the choice of which depends upon the problem under consideration. When the equations governing the deformation are expanded in terms of this parameter, the coefficients of each successive power of ϵ furnish a set of relations for the determination of the corresponding terms in the expansions for the stress and displacement functions. In the present paper attention is limited to terms of the first and second orders.

The reduction of the theory to two-dimensional form makes possible a formulation in complex variable notation similar to that of the classical theory of elasticity (see, for example, Muskhelishvili 1953; Green & Zerna 1954). Explicit expressions for the stress and displacement functions can then be obtained in terms of complex potential functions, which are chosen to satisfy the boundary conditions in a given problem, two additional functions being introduced for each succeeding stage of the approximation process. The resulting expressions are similar in form to those derived by Adkins *et al.* in developing the theory of finite plane strain for incompressible materials, but with different values for the constant coefficients. In considering finite deformations, the complex co-ordinate system may be related either to points in the deformed body or to points in the undeformed body, the choice for any particular problem depending upon the nature of the boundary conditions. For convenience, the theory is developed in terms of complex co-ordinates in the deformed body, the corresponding formulae for co-ordinates in the undeformed body being obtained by a simple change of independent variable.

The similarity of the results for plane stress to those obtained for incompressible materials in plane strain suggests naturally the possibility of formulating in more general terms the second-order theory for two-dimensional problems. The approximate theory for compressible materials in plane strain is therefore developed in § 9; in the final section of the paper the results previously obtained are combined to yield general formulae for the determination of second-order solutions of two-dimensional problems in elasticity. These formulae express the stress and displacement functions in terms of complex potential functions, but the constant coefficients are left arbitrary. The complex potential functions may then be chosen, using these equations, to satisfy a prescribed set of boundary conditions over given contours, and the stress and displacement components evaluated in general terms. The results for plane stress or plane strain can then be obtained as special cases of the more general solution by choosing the appropriate values for the constants. Moreover, by a suitable choice of constants, the contours over which the boundary conditions are specified can form the boundaries either of the deformed body or of the undeformed body. Since the results for an incompressible material can be obtained by a limiting process from those for a compressible material, the single general solution can be made to yield, as special cases, the results for eight associated problems.

From a detailed examination of the equations for plane stress and plane strain, it is shown that the constants in the general solution may be expressed as functions of the elastic constants of the material together with two additional parameters. One of these parameters is employed to differentiate between plane stress and plane strain, while the other is chosen to distinguish between co-ordinates in the undeformed body and in the deformed body.

2. Notation and formulae

With slight modifications* we use the notation of Green & Zerna (1950) and Green & Shield (1950, 1951). The points of an unstrained and unstressed body at rest at time $t = 0$ are defined by a system of rectangular Cartesian co-ordinates x_i or by a general curvilinear

* See *Theoretical elasticity* by Green & Zerna (1954).

system of co-ordinates θ_i. The points of the deformed body may also be defined by a set of rectangular Cartesian co-ordinates y_i, and in the present paper we shall take the x_i-axes and y_i-axes to coincide. The curvilinear co-ordinates θ_i move with the body as it is deformed and form a curvilinear system in the strained body at time t. The covariant and contravariant metric tensors for the co-ordinate system θ_i in the unstrained body are denoted by g_{ij} and g^{ij} respectively, and for the co-ordinate system in the strained body, at time t, the corresponding metric tensors are G_{ij} and G^{ij} respectively. We write

$$g = |\, g_{ij}|, \quad G = |\, G_{ij}\,|, \tag{2·1}$$

latin indices taking the values 1, 2, 3.

For a homogeneous, isotropic, elastic material the strain-energy function W, measured per unit volume of the unstrained body, may be regarded as a function of three strain invariants I_1, I_2, I_3 given by

$$I_1 = g^{ij}G_{ij}, \quad I_2 = I_3 g_{ij}G^{ij}, \quad I_3 = G/g, \tag{2·2}$$

so that

$$W = W(I_1, I_2, I_3). \tag{2·3}$$

The contravariant stress tensor τ^{ij}, measured per unit area of the strained body, and referred to co-ordinates in the strained body may be expressed in the form

$$\tau^{ij} = g^{ij}\Phi + B^{ij}\Psi + G^{ij}p, \tag{2·4}$$

where

$$\Phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}, \tag{2·5}$$

$$B^{ij} = g^{ij}I_1 - g^{ir}g^{js}G_{rs} = \frac{1}{g}\,\mathrm{e}^{irm}\,\mathrm{e}^{jsn}g_{rs}G_{mn}, \tag{2·6}$$

and e^{irm} is equal to $+1$ or -1 according as i, r, m is an even or odd permutation of 1, 2, 3, and equal to 0 otherwise.

If $\mathbf{t}$ is the stress vector associated with a surface in the deformed body whose unit normal $\mathbf{u}$ is given by

$$\mathbf{u} = u_i\mathbf{G}^i, \tag{2·7}$$

then

$$\mathbf{t} = \frac{u_i\mathbf{T}_i}{\sqrt{G}} = u_i\tau^{ij}\mathbf{G}_j = \sum_i u_i\mathbf{t}_i\sqrt{G^{ii}}, \tag{2·8}$$

where

$$\mathbf{T}_i = \sqrt{(GG^{ii})}\,\mathbf{t}_i = \sqrt{(G)}\,\tau^{ij}\mathbf{G}_j. \tag{2·9}$$

$\mathbf{G}_j$, $\mathbf{G}^j$ are the covariant and contravariant base vectors in the deformed body, and $\mathbf{t}_i$ denotes the stress vector associated with the surface $\theta_i =$ constant.

With this notation the equations of equilibrium in the absence of body forces may be written in the alternative forms

$$\mathbf{T}_{i,i} = 0, \tag{2·10}$$

$$\tau^{ij}\,\|_i = 0, \tag{2·11}$$

where in (2·10) the comma denotes partial differentiation with respect to θ_i, and in (2·11) the double line denotes covariant differentiation with respect to the deformed body, that is, with respect to θ_i and the metric tensor components G_{ij}, G^{ij}.

PLANE STRESS

3. Stress resultants and loads

In this section the development of the theory is similar to that given by Green & Zerna (1954) for the classical theory of plates. We suppose the unstrained body to be a plate of homogeneous isotropic elastic material bounded by the plane surfaces $x_3 = \pm h_0$, where h_0 is a constant, although the results of §§ 3, 4 are also valid for an aeolotropic plate which has symmetry about the plane $x_3 = 0$. This plate undergoes a finite deformation symmetrical about the middle plane $x_3 = 0$, which thus becomes the middle plane $y_3 = 0$ in the deformed state. The major surfaces of the plate after deformation are given by $y_3 = \pm h$, where h is, in general, a function of y_1, y_2. We choose the curvilinear co-ordinate system θ_i so that

$$y_3 = \theta_3, \quad y_\alpha = y_\alpha(\theta_1, \theta_2, t), \tag{3·1}$$

greek indices taking the values 1, 2. It follows that

$$G_{ij} = \begin{pmatrix} A_{11}, & A_{12}, & 0 \\ A_{12}, & A_{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} A^{11}, & A^{12}, & 0 \\ A^{12}, & A^{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G = A, \tag{3·2}$$

with
$$A = |A_{\alpha\beta}|, \quad A^{\alpha\rho}A_{\rho\beta} = \delta^\alpha_\beta, \tag{3·3}$$

where $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with co-ordinates θ_α in the middle plane $y_3 = 0$ of the deformed plate, and δ^α_β is the Kronecker delta.

The force acting on an element of area of the co-ordinate surface $\theta_1 =$ constant in the deformed body is $\mathbf{T}_1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3$, and the length of the corresponding line element of the middle plane $y_3 = 0$ is

$$\surd(A_{22})\,\mathrm{d}\theta^2 = \surd(AA^{11})\,\mathrm{d}\theta^2.$$

Similar considerations apply for the other co-ordinate surface $\theta_2 =$ constant. The stress across either of the surfaces $\theta_\alpha =$ constant may therefore be replaced by a physical stress resultant $\mathbf{n}_\alpha$, measured per unit length of the curve $\theta_\alpha =$ constant in the plane $y_3 = 0$, where

$$\mathbf{n}_\alpha = \frac{\mathbf{N}_\alpha}{\surd(AA^{\alpha\alpha})}, \quad \mathbf{N}_\alpha = \int_{-h}^{h} \mathbf{T}_\alpha\,\mathrm{d}y_3, \tag{3·4}$$

and we recall that h is a function of y_1, y_2 or θ_1, θ_2. Since the deformation is symmetrical about $y_3 = 0$, the corresponding stress couples are zero. From (2·9)

$$\mathbf{T}_\alpha = \surd(A)\,\tau^{\alpha j}\mathbf{G}_j,$$

so that from (3·4) we may write

$$\mathbf{n}_\alpha \surd A^{\alpha\alpha} = n^{\alpha\rho}\mathbf{G}_\rho + q^\alpha \mathbf{G}_3, \quad \mathbf{N}_\alpha = N^{\alpha\rho}\mathbf{G}_\rho + Q^\alpha\mathbf{G}_3, \tag{3·5}$$

where
$$N^{\alpha\rho} = n^{\alpha\rho}\surd A, \quad Q^\alpha = q^\alpha \surd A, \tag{3·6}$$

and
$$n^{\alpha\rho} = \int_{-h}^{h} \tau^{\alpha\rho}\,\mathrm{d}y_3, \quad q^\alpha = \int_{-h}^{h} \tau^{\alpha 3}\,\mathrm{d}y_3. \tag{3·7}$$

Since the deformation, and consequently the stress distribution, is symmetrical about the plane $y_3 = 0$, it follows that $q^\alpha = 0$.

The stress resultant $\mathbf{n}$ per unit length of a line drawn in the middle plane $y_3 = 0$ of the deformed plate, whose unit normal in that plane is

$$\mathbf{u} = u_\alpha \mathbf{G}^\alpha,$$

is given by

$$\mathbf{n} = \int_{-h}^{h} \mathbf{t}\, dy_3, \tag{3·8}$$

so that, from (2·8), (3·4) and (3·5),

$$\mathbf{n} = \frac{u_\alpha}{\sqrt{A}} \int_{-h}^{h} \mathbf{T}_\alpha\, dy_3 = \frac{u_\alpha \mathbf{N}_\alpha}{\sqrt{A}} = \sum_1^2 u_\alpha \mathbf{n}_\alpha \sqrt{A^{\alpha\alpha}} = u_\alpha (n^{\alpha\rho}\mathbf{G}_\rho + q^\alpha \mathbf{G}_3). \tag{3·9}$$

The functions defined in (3·7) are (plane) surface tensors. The components of the symmetrical contravariant tensor $n^{\alpha\rho}$ and the components of the contravariant tensor q^α are called stress resultants and shearing forces respectively. Mixed and covariant tensors n^α_β, $n_{\alpha\beta}$, q_α may be formed with the help of the metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$. In order to find the physical components of $\mathbf{n}_\alpha$ we express these vectors in terms of unit base vectors along the co-ordinate curves $\theta_\alpha =$ constant. The physical stress resultants and shearing forces are denoted by $n_{(\alpha\beta)}$, $q_{(\alpha)}$ respectively, the bracket indicating that these quantities are not tensors. Thus we have

$$\mathbf{n}_\alpha = n_{(\alpha 1)} \frac{\mathbf{G}_1}{\sqrt{A_{11}}} + n_{(\alpha 2)} \frac{\mathbf{G}_2}{\sqrt{A_{22}}} + q_{(\alpha)} \mathbf{G}_3, \tag{3·10}$$

and comparison of this with (3·5) yields

$$n_{(\alpha\beta)} = n^{\alpha\beta} \sqrt{(A_{\beta\beta}/A^{\alpha\alpha})}, \quad q_{(\alpha)} = q^\alpha / \sqrt{A^{\alpha\alpha}}. \tag{3·11}$$

We now consider the external forces acting on the major surfaces of the deformed plate. The covariant components u_i of the unit normal to the surfaces $y_3 = \pm h(\theta_1, \theta_2)$ referred to the base vectors $\mathbf{G}^i$ are

$$(u_1, u_2, u_3) = k\left(-\frac{\partial y_3}{\partial \theta_1}, -\frac{\partial y_3}{\partial \theta_2}, 1\right), \tag{3·12}$$

where, remembering (3·2)

$$k = (A^{\alpha\beta} h_{,\alpha} h_{,\beta} + 1)^{-\frac{1}{2}}, \tag{3·13}$$

and at these surfaces, from (2·8),

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{\sqrt{A}} = \frac{k}{\sqrt{A}} (\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}). \tag{3·14}$$

The stress vector $\mathbf{t}$ is measured per unit area of the surfaces $y_3 = \pm h$. But

$$dS = (u_3^{-1})\, dS_3 = (\sqrt{A}/k)\, d\theta^1 d\theta^2,$$

where dS, dS_3 are corresponding elements of one of the major surfaces of the plate and of the middle plane respectively. We can thus replace $\mathbf{t}$ by $\mathbf{t}/k$ measured per unit area of the middle plane $y_3 = 0$. We now replace the surface forces by a resultant force $\mathbf{l}$ measured per unit area of this plane, where

$$\mathbf{l} = [\mathbf{t}/k]_{y_3 = h} - [\mathbf{t}/k]_{y_3 = -h} = [\mathbf{t}]^h_{-h}/k. \tag{3·15}$$

If we introduce the vector $\mathbf{L}$ where

$$\mathbf{L} = \mathbf{l}\sqrt{A}, \tag{3·16}$$

then, from (3·14),

$$\mathbf{L} = (\sqrt{A}/k)\, [\mathbf{t}]^h_{-h} = [\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}]^h_{-h}. \tag{3·17}$$

Remembering (2·9) we may now write **l**, **L** in the forms

$$\mathbf{l} = l^\alpha \mathbf{G}_\alpha + l\mathbf{G}_3, \quad \mathbf{L} = L^\alpha \mathbf{G}_\alpha + L\mathbf{G}_3, \tag{3·18}$$

where

$$\left.\begin{aligned} L^\alpha &= l^\alpha \sqrt{A}, \quad L = l\sqrt{A}, \\ l^\alpha &= [\tau^{\alpha 3} - \tau^{\alpha\beta} y_{3,\beta}]_{-h}^{h}, \quad l = [\tau^{33} - \tau^{3\beta} y_{3,\beta}]_{-h}^{h}. \end{aligned}\right\} \tag{3·19}$$

These relations may be simplified by observing that since the deformation is symmetrical about the plane $y_3 = 0$, l and L are zero. Thus (3·18) and (3·19) yield

$$\left.\begin{aligned} \mathbf{l} = l^\alpha \mathbf{G}_\alpha, \quad \mathbf{L} = L^\alpha \mathbf{G}_\alpha, \\ [\tau^{33} - \tau^{3\beta} y_{3,\beta}]_{-h}^{h} = 0. \end{aligned}\right\} \tag{3·20}$$

4. Equations of equilibrium: Airy's stress function

If we integrate (2·10) through the thickness of the plate we obtain

$$\int_{-h}^{h} \mathbf{T}_{\alpha,\alpha}\, dy_3 + [\mathbf{T}_3]_{-h}^{h} = 0. \tag{4·1}$$

But, from (3·4),

$$\mathbf{N}_{\alpha,\alpha} = \frac{\partial}{\partial \theta^\alpha} \int_{-h}^{h} \mathbf{T}_\alpha\, dy_3 = \int_{-h}^{h} \mathbf{T}_{\alpha,\alpha}\, dy_3 + [\mathbf{T}_\alpha y_{3,\alpha}]_{-h}^{h},$$

so that (4·1) becomes

$$\mathbf{N}_{\alpha,\alpha} + [\mathbf{T}_3 - \mathbf{T}_\alpha y_{3,\alpha}]_{-h}^{h} = 0,$$

or

$$\mathbf{N}_{\alpha,\alpha} + \mathbf{L} = 0, \tag{4·2}$$

if we use (3·17). Combining this with (3·5), (3·6), (3·19) and (3·20) we have

$$n^{\alpha\beta} \|_\alpha + l^\beta = 0, \tag{4·3}$$

where the double line denotes covariant differentiation with respect to the plane variables θ_α in the deformed body, using Christoffel symbols formed from the metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$. Since q^α and l are zero, the third equation of equilibrium is automatically satisfied.

We shall, from now on, assume that the major surfaces of the plate are free from applied forces so that $\mathbf{t} = 0$ when $y_3 = \pm h$. Then, from (3·16) and (3·17), we see that **L** and **l** are zero and, remembering (3·20), the equations of equilibrium (4·2), (4·3) reduce to

$$\mathbf{N}_{\alpha,\alpha} = 0 \quad \text{or} \quad n^{\alpha\beta} \|_\alpha = 0. \tag{4·4}$$

Equations (4·4) and (3·5), with $q^\alpha = 0$, are similar in form to the corresponding equations for $\mathbf{T}_\alpha$ and $\tau^{\alpha\beta}$ obtained by Adkins *et al.* (1953) for plane strain. The results there obtained may therefore be applied to express the stress resultants and applied forces and couples in terms of an Airy stress function ϕ. Thus

$$\left.\begin{aligned} \mathbf{N}_\alpha &= \sqrt{(A)}\, \epsilon^{\gamma\alpha} \boldsymbol{\chi}_{,\gamma} = \sqrt{(A)}\, \epsilon^{\gamma\alpha} \epsilon^{\rho\beta} \phi \|_{\gamma\rho} \mathbf{G}_\beta, \\ n^{\alpha\beta} &= \epsilon^{\alpha\gamma} \epsilon^{\beta\rho} \phi \|_{\gamma\rho}, \end{aligned}\right\} \tag{4·5}$$

and

or

$$\phi \|_{\alpha\beta} = \epsilon_{\alpha\gamma} \epsilon_{\beta\rho} n^{\gamma\rho} = (A/a)\, ({}_0\epsilon_{\alpha\gamma})\, ({}_0\epsilon_{\beta\rho})\, n^{\gamma\rho}, \tag{4·6}$$

where $\boldsymbol{\chi}$ is a vector in the plane $y_3 = 0$, ϕ is a scalar invariant function of θ_1 and θ_2,

$${}_0\epsilon^{\alpha\beta} \sqrt{a} = {}_0\epsilon_{\alpha\beta}/\sqrt{a} = \epsilon^{\alpha\beta} \sqrt{A} = \epsilon_{\alpha\beta}/\sqrt{A} = \mathrm{e}_{\alpha\beta 3}, \tag{4·7}$$

and

$$\epsilon_{\alpha\rho} \epsilon^{\alpha\lambda} = \delta^\lambda_\rho.$$

The double line again indicates covariant differentiation with respect to the plane $y_3 = 0$ of the deformed body, the order of differentiation being immaterial since the Riemann-Christoffel tensor in the plane vanishes.

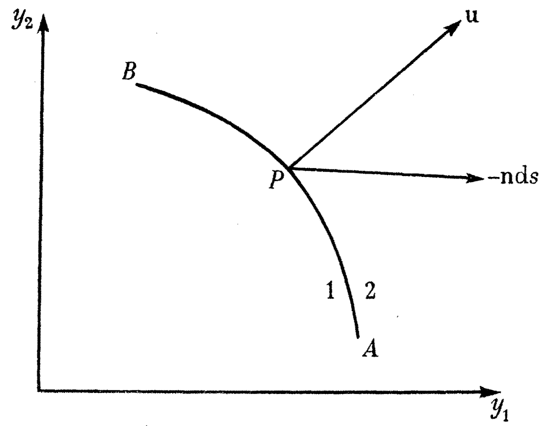

FIGURE 1

Let AP be an arc of a curve AB in the middle plane $y_3 = 0$ of the deformed body (figure 1). By an analysis similar to that used for plane strain we may obtain the resultant force across a surface in the deformed body formed by normals to $y_3 = 0$ along AB. Denoting an element of AP by ds and making use of (3·9) and (4·5), we obtain for the total force $\mathbf{F}$ exerted by the region 1 on the region 2, across the arc AP,

$$\mathbf{F} = -\int_A^P \mathbf{n}\, ds = \boldsymbol{\chi} = \epsilon^{\rho\beta}\phi_{,\rho}\mathbf{G}_\beta, \tag{4·8}$$

apart from an arbitrary constant vector which may be absorbed into $\boldsymbol{\chi}$ without loss of generality. Similarly, the total moment about the y_3-axis of the forces exerted by the region 1 on the region 2 is given by

$$\mathbf{M} = \int_A^P [\mathbf{R} \wedge \boldsymbol{\chi}_{,\beta}] \frac{d\theta^\beta}{ds} ds = (R^\alpha \phi_{,\alpha} - \phi)\, \mathbf{G}^3, \tag{4·9}$$

apart from an arbitrary constant vector which may again be absorbed into $\phi\mathbf{G}^3$ without affecting the stresses. In (4·9)

$$\mathbf{R} = R^\alpha \mathbf{G}_\alpha = R_\alpha \mathbf{G}^\alpha \tag{4·10}$$

is the position vector of a point on the curve AB with respect to the origin of the y_i-axes. Equation (4·9) thus represents a couple of magnitude

$$M = R^\alpha \phi_{,\alpha} - \phi \tag{4·11}$$

about the y_3-axis.

If AB is a boundary curve of the plate which is entirely free from applied forces, (4·8) and (4·11) yield the conditions

$$\boldsymbol{\chi} = 0,$$

or

$$\phi_{,1} = 0, \quad \phi_{,2} = 0, \tag{4·12}$$

at all points of AB.

5. Stress-strain relations

In the equations so far derived, no assumptions have been made regarding the thickness of the plate. We now confine our attention to plates which are thin, and write approximately

$$x_\alpha = x_\alpha(\theta_1, \theta_2), \quad x_3 = y_3/\lambda = \theta_3/\lambda, \tag{5·1}$$

where λ is a scalar invariant function of θ_1, θ_2. The metric tensors g_{ij}, g^{ij} now take the approximate forms

$$\left.\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta}, \quad g_{33} = 1/\lambda^2, \\ g^{\alpha\beta} &= a^{\alpha\beta}, \quad g^{33} = \lambda^2, \\ g &= a/\lambda^2, \quad a = |\, a_{\alpha\beta} \,|, \end{aligned}\right\} \tag{5·2}$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear co-ordinates θ_α in the plane $x_3 = 0$ of the undeformed body.

From (2·2), (3·2) and (5·2) the strain invariants are given by

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{\alpha\beta} A_{\alpha\beta}, \\ I_2 &= \lambda^2 (A/a)\, a_{\alpha\beta} A^{\alpha\beta} + A/a, \\ I_3 &= \lambda^2 A/a, \end{aligned}\right\} \tag{5·3}$$

approximately. Also, remembering (4·7), we have $a_{\alpha\beta} A^{\alpha\beta} A = a^{\alpha\beta} A_{\alpha\beta} a$ and hence

$$I_3 - \lambda^2 I_2 + \lambda^4 I_1 - \lambda^6 = 0. \tag{5·4}$$

These results bear a formal resemblance to the corresponding relations obtained by Adkins *et al.* (1953) for plane strain, but λ is no longer constant.

The tensor B^{ij} may be calculated from (2·6), (3·2) and (5·2), and is approximately

$$\left.\begin{aligned} B^{\alpha\beta} &= \lambda^2 a^{\alpha\beta} + A A^{\alpha\beta}/a, \\ B^{33} &= \lambda^2 (I_1 - \lambda^2). \end{aligned}\right\} \tag{5·5}$$

From (2·4), (3·2), (5·2) and (5·5) we obtain for the components of the stress tensor

$$\left.\begin{aligned} \tau^{\alpha\beta} &= (\Phi + \lambda^2 \Psi)\, a^{\alpha\beta} + (\Psi A/a + p)\, A^{\alpha\beta}, \\ \tau^{33} &= \lambda^2 \Phi + \lambda^2 (I_1 - \lambda^2)\, \Psi + p. \end{aligned}\right\} \tag{5·6}$$

Since the major surfaces of the plate are free from applied forces, from (3·15) and (3·19) we have, at $y_3 = \pm h$,

$$\tau^{\alpha 3} - \tau^{\alpha\beta} y_{3,\beta} = 0, \quad \tau^{33} - \tau^{3\alpha} y_{3,\alpha} = 0,$$

from which, by eliminating $\tau^{\alpha 3}$, we obtain

$$\tau^{33} - \tau^{\alpha\beta} y_{3,\alpha} y_{3,\beta} = 0. \tag{5·7}$$

If the thickness of the plate is sufficiently small and if $h_{3,\alpha}$ is of the same order of magnitude as h, it is evident from (5·7) that at $y_3 = \pm h$, τ^{33} is small compared with the stresses $\tau^{\alpha\beta}$. We therefore assume τ^{33} to be negligible throughout the plate, and from the last of equations (5·6) we then have approximately

$$\lambda^2 \Phi + \lambda^2 (I_1 - \lambda^2)\, \Psi + p = 0. \tag{5·8}$$

Eliminating p between (5·8) and the first equation of (5·6) we obtain

$$\tau^{\alpha\beta} = (\Phi + \lambda^2\Psi)\, a^{\alpha\beta} + \{(A/a + \lambda^4 - \lambda^2 I_1)\,\Psi - \lambda^2\Phi\}\, A^{\alpha\beta}. \tag{5·9}$$

From (5·3) it is evident that the invariants, and hence also Φ and Ψ, are independent of y_3 $(=\theta_3)$. Equation (3·7) thus yields

$$n^{\alpha\beta} = 2h\tau^{\alpha\beta} = 2\lambda h_0\,\tau^{\alpha\beta}. \tag{5·10}$$

From (4·6), (5·9) and (5·10)

$$\phi\,\|_{\alpha\beta} = H a_{\alpha\beta} + K A_{\alpha\beta}, \tag{5·11}$$

where, remembering (2·5) and (5·3), we have

$$\left.\begin{aligned} H &= 2h_0\lambda\frac{A}{a}\{\Phi + \lambda^2\Psi\} = 4h_0\frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right\},\\ K &= 2h_0\lambda\left\{\left(\frac{A}{a} + \lambda^4 - \lambda^2 I_1\right)\Psi - \lambda^2\Phi\right\}\\ &= -4h_0\frac{\lambda}{\sqrt{I_3}}\left\{\lambda^2\frac{\partial W}{\partial I_1} + \left(\lambda^2 I_1 - \lambda^4 - \frac{I_3}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right\}.\end{aligned}\right\} \tag{5·12}$$

For an incompressible material

$$I_3 = \lambda^2 A/a = 1, \tag{5·13}$$

and W becomes a function of I_1 and I_2 only. Equations (5·12) then reduce to

$$\left.\begin{aligned} H &= \frac{4h_0}{\lambda}\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right),\\ K &= -4h_0\lambda\left\{\lambda^2\frac{\partial W}{\partial I_1} + \left(\lambda^2 I_1 - \lambda^4 - \frac{1}{\lambda^2}\right)\frac{\partial W}{\partial I_2}\right\}.\end{aligned}\right\} \tag{5·14}$$

For compressible materials it is convenient to express H and K in terms of three different, mutually independent invariants J_1, J_2, J_3 defined by

$$\left.\begin{aligned} J_1 &= I_1 - 3,\\ J_2 &= I_2 - 2I_1 + 3,\\ J_3 &= I_3 - I_2 + I_1 - 1.\end{aligned}\right\} \tag{5·15}$$

The invariants J_1, J_2, J_3 have been employed by Rivlin (1953) and have the advantage that for small deformations they are of the first, second and third orders of smallness respectively. From (5·15) we have

$$\left.\begin{aligned} \frac{\partial W}{\partial I_1} &= \frac{\partial W}{\partial J_1} - 2\frac{\partial W}{\partial J_2} + \frac{\partial W}{\partial J_3},\\ \frac{\partial W}{\partial I_2} &= \frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3},\\ \frac{\partial W}{\partial I_3} &= \frac{\partial W}{\partial J_3}.\end{aligned}\right\} \tag{5·16}$$

Equations (5·4) and (5·8) then yield

$$J_3 + (1-\lambda^2)\,J_2 + (1-\lambda^2)^2\,J_1 + (1-\lambda^2)^3 = 0, \tag{5·17}$$

and

$$\lambda^2\frac{\partial W}{\partial J_1} + \lambda^2(J_1 + 1 - \lambda^2)\frac{\partial W}{\partial J_2} + \{(1-\lambda^2)\,(J_1 + 1 - \lambda^2) + J_2 + J_3\}\frac{\partial W}{\partial J_3} = 0, \tag{5·18}$$

respectively. Expressions corresponding to (5·12) may be derived for H and K, but in place of (5·9) we now make use of the first of (5·6) from which p has not been eliminated. Thus we obtain

$$\left.\begin{aligned} H &= 4h_0 \frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}, \\ K &= 4h_0 \frac{\sqrt{I_3}}{\lambda}\left\{\frac{\partial W}{\partial J_2}+(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}. \end{aligned}\right\} \tag{5·19}$$

For a compressible material these expressions are equivalent to (5·12) by virtue of (5·16), (5·17) and (5·18).

6. Formulation in terms of complex variables

With the simplifying assumptions of the previous section, and the consequent reduction of the theory to two-dimensional form, it becomes possible to employ complex variable techniques. The procedure followed in this and subsequent sections is therefore similar to that used by Adkins *et al.* (1953) in the treatment of finite plane strain. The presence of the parameter λ, however, which is now a function of the co-ordinates, renders the equations obtained more complicated in form.

For finite deformations, the complex co-ordinate reference frame may be related to points in the undeformed body or in the deformed body, and the relevant equations for either co-ordinate system may be derived by an appropriate choice for the moving system of co-ordinates θ_α in the relations of the preceding sections. Since the resulting expressions are simpler in form for complex co-ordinates in the deformed body, we shall consider this case first, and from the results thus obtained, derive the corresponding formulae for complex co-ordinates in the undeformed body by a simple change of independent variable.

We thus introduce complex co-ordinates $(\zeta, \bar{\zeta})$, $(z, \bar{z})$ in the undeformed body and in the deformed body respectively by means of the relations

$$\left.\begin{aligned} \zeta &= x_1 + \mathrm{i}x_2, \quad \bar{\zeta} = x_1 - \mathrm{i}x_2, \\ z &= y_1 + \mathrm{i}y_2, \quad \bar{z} = y_1 - \mathrm{i}y_2. \end{aligned}\right\} \tag{6·1}$$

If the components of displacement referred to the x_α-axes are (u, v), the complex displacement function D is defined by

$$D = u + \mathrm{i}v, \quad \bar{D} = u - \mathrm{i}v, \tag{6·2}$$

and since the x_α-axes and y_α-axes coincide, we have

$$z = \zeta + D, \quad \bar{z} = \bar{\zeta} + \bar{D}. \tag{6·3}$$

If we denote covariant and contravariant base vectors in the system of complex co-ordinates $(z, \bar{z})$ by $\mathbf{A}_\alpha$ and $\mathbf{A}^\alpha$ respectively, the position vector $\mathbf{R}$ of a point of the deformed body, which is given by (4·10), may be written

$$\mathbf{R} = z^\alpha \mathbf{A}_\alpha = z_\alpha \mathbf{A}^\alpha.$$

By tensor transformations

$$z^1 = \frac{\partial z}{\partial y_1} y_1 + \frac{\partial z}{\partial y_2} y_2 = y_1 + \mathrm{i}y_2 = z,$$

$$z^2 = \frac{\partial \bar{z}}{\partial y_1} y_1 + \frac{\partial \bar{z}}{\partial y_2} y_2 = y_1 - \mathrm{i}y_2 = \bar{z},$$

so that the complex co-ordinates $(z, \bar{z})$ may be denoted by z^α.

We now take the moving system of co-ordinates θ_α to coincide with the co-ordinates $(z, \bar{z})$ so that

$$\theta_1 = z, \quad \theta_2 = \bar{z}. \tag{6·4}$$

The metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$ then have the values

$$\left.\begin{aligned} A_{12} &= \tfrac{1}{2}, \quad A_{11} = A_{22} = 0, \quad \sqrt{A} = \tfrac{1}{2}\mathrm{i}, \\ A^{12} &= 2, \quad A^{11} = A^{22} = 0. \end{aligned}\right\} \tag{6·5}$$

Remembering (5·2), the strain invariants (5·3) reduce to

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{12}, \\ I_2 &= -\lambda^2 a_{12}/a - 1/(4a), \\ I_3 &= -\lambda^2/(4a), \end{aligned}\right\} \tag{6·6}$$

where, from (5·1), (5·2), (6·1), (6·3) and (6·4), we have

$$\sqrt{a} = \frac{\partial(x_1, x_2)}{\partial(\theta_1, \theta_2)} = \frac{\mathrm{i}}{2}\left(1 - \frac{\partial D}{\partial z} - \frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} - \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right) = \frac{\mathrm{i}\lambda}{2\sqrt{I_3}}, \tag{6·7}$$

$$\left.\begin{aligned} a_{11} &= \bar{a}_{22} = \left(\frac{\partial x_1}{\partial z}\right)^2 + \left(\frac{\partial x_2}{\partial z}\right)^2 = \frac{\partial \overline{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right), \\ a_{12} &= \frac{1}{2}\left\{1 - \frac{\partial D}{\partial z} - \frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\} \\ &= \frac{\lambda}{2\sqrt{I_3}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ a^{11} &= \bar{a}^{22} = a_{11}/a, \quad a^{12} = -a_{12}/a. \end{aligned}\right\} \tag{6·8}$$

A bar over a function indicates the complex conjugate of that function, and we have used (6·7) to simplify a_{12}. Introducing these results into (6·6) and using (5·15) we obtain

$$\left.\begin{aligned} J_1 &= \lambda^2 - 3 - \frac{a_{12}}{a} = \lambda^2 + 2\frac{\sqrt{I_3}}{\lambda} + 4\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z} - 3, \\ J_2 &= 3 - 2\lambda^2 - (\lambda^2 - 2)\frac{a_{12}}{a} - \frac{1}{4a} \\ &= 3 - 2\lambda^2 + \frac{I_3}{\lambda^2} + 2(\lambda^2 - 2)\left\{\frac{\sqrt{I_3}}{\lambda} + 2\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\}, \\ J_3 &= (\lambda^2 - 1)\left(1 + \frac{a_{12}}{a} - \frac{1}{4a}\right) \\ &= (\lambda^2 - 1)\left\{\left(1 - \frac{\sqrt{I_3}}{\lambda}\right)^2 - 4\frac{I_3}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}\right\}. \end{aligned}\right\} \tag{6·9}$$

If the body is incompressible so that $I_3 = 1$, we have from (6·6), (6·7) and (6·8)

$$\sqrt{a} = \mathrm{i}\lambda/2,$$

or

$$1 - \lambda = \frac{\partial D}{\partial z} + \frac{\partial \overline{D}}{\partial \bar{z}} - \frac{\partial D}{\partial z}\frac{\partial \overline{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \tag{6·10}$$

and

$$\left.\begin{aligned} a_{12} &= \tfrac{1}{2}\lambda + \frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ I_1 &= \lambda^2 + \frac{2}{\lambda} + \frac{4}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}, \\ I_2 &= \frac{1}{\lambda^2} + 2\lambda + 4\frac{\partial D}{\partial \bar{z}}\frac{\partial \overline{D}}{\partial z}. \end{aligned}\right\} \tag{6·11}$$

Since the components (6·5) of the metric tensor of the deformed body are constants, the corresponding Christoffel symbols are zero and therefore covariant differentiation in the deformed body reduces to partial differentiation. The equations of equilibrium (5·11) thus reduce to

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= Ha_{11},\\ \frac{\partial^2\phi}{\partial z\,\partial\bar{z}} &= Ha_{12}+\tfrac{1}{2}K,\end{aligned}\right\} \tag{6·12}$$

together with the complex conjugate of the first of these equations.

For a compressible material H and K are given by (5·19), and (6·12) then becomes

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= 4h_0\left(\frac{\sqrt{I_3}}{\lambda}\right)\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial\bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\,\partial\bar{z}} &= 2h_0\left\{\frac{\partial W}{\partial J_1}+\left(\frac{\sqrt{I_3}}{\lambda}+\lambda^2-2\right)\frac{\partial W}{\partial J_2}+(\lambda^2-1)\left(\frac{\sqrt{I_3}}{\lambda}-1\right)\frac{\partial W}{\partial J_3}\right\}\\ &\quad +4h_0\left(\frac{\sqrt{I_3}}{\lambda}\right)\left\{\frac{\partial W}{\partial J_1}+(\lambda^2-2)\frac{\partial W}{\partial J_2}-(\lambda^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial D}{\partial\bar{z}}\frac{\partial\bar{D}}{\partial z}.\end{aligned}\right\} \tag{6·13}$$

Also, from (6·9) and (5·18), we have

$$\frac{\partial W}{\partial J_1}+2\left(\frac{\sqrt{I_3}}{\lambda}-1\right)\frac{\partial W}{\partial J_2}+\left(\frac{\sqrt{I_3}}{\lambda}-1\right)^2\frac{\partial W}{\partial J_3}+4\frac{I_3}{\lambda^2}\left(\frac{\partial W}{\partial J_2}-\frac{\partial W}{\partial J_3}\right)\frac{\partial D}{\partial\bar{z}}\frac{\partial\bar{D}}{\partial z}=0. \tag{6·14}$$

Remembering (6·7) and (6·9), the relations (6·13) and (6·14) yield four equations for the determination of ϕ, D, $\bar{D}$ and λ.

The corresponding equations for an incompressible material are obtained by introducing (5·14) and (6·11) into (6·12). Thus

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= \frac{4h_0}{\lambda}\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\frac{\partial\bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\,\partial\bar{z}} &= 2h_0\left\{(1-\lambda^3)\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)+2\left(\frac{1}{\lambda}\frac{\partial W}{\partial I_1}-\lambda\frac{\partial W}{\partial I_2}\right)\frac{\partial D}{\partial\bar{z}}\frac{\partial\bar{D}}{\partial z}\right\}.\end{aligned}\right\} \tag{6·15}$$

These equations, together with the incompressibility condition (6·10), are again sufficient to determine ϕ, D, $\bar{D}$ and λ.

The theory may be formulated similarly in terms of complex co-ordinates in the undeformed body by choosing the moving system of co-ordinates θ_α to coincide with the complex co-ordinate system $(\zeta, \bar{\zeta})$. Alternatively, by making use of (6·3) and (6·7) we may change the independent variables in equations (6·9) to (6·15). Thus for a compressible material we may write

$$\left.\begin{aligned}\frac{\partial}{\partial z} &= \frac{\lambda}{\sqrt{I_3}}\left\{\left(1+\frac{\partial\bar{D}}{\partial\bar{\zeta}}\right)\frac{\partial}{\partial\zeta}-\frac{\partial\bar{D}}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}\right\},\\ \frac{\partial}{\partial\bar{z}} &= \frac{\lambda}{\sqrt{I_3}}\left\{-\frac{\partial D}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}+\left(1+\frac{\partial D}{\partial\zeta}\right)\frac{\partial}{\partial\bar{\zeta}}\right\},\end{aligned}\right\} \tag{6·16}$$

and for an incompressible material we may put $I_3 = 1$ in these expressions. The resulting equations will not, however, be required for the approximate theory developed in §§7 to 10.

Denoting the stress resultants referred to complex co-ordinates in the deformed body by $n^{\alpha\beta}$, we have from (4·5)

$$n^{11} = \overline{n}^{22} = -4\frac{\partial^2\phi}{\partial \overline{z}^2}, \quad n^{12} = 4\frac{\partial^2\phi}{\partial z\,\partial \overline{z}}. \tag{6·17}$$

Employing (6·3) and (6·1), we may now, by simple tensor transformations, obtain expressions for the stress resultants referred to the complex co-ordinate system $(\zeta, \overline{\zeta})$, or to the real co-ordinate systems x_α, y_α, analogous to those obtained for the stress components in the corresponding theory of finite plane strain.

Also, if the resultant force $\mathbf{F}$ across any arc AP of a curve in the deformed plate has components (X, Y) along the y_1-, y_2-axes respectively, then a simple tensor transformation gives

$$\mathbf{F} = (X+\mathrm{i}Y)\,\mathbf{A}_1 + (X-\mathrm{i}Y)\,\mathbf{A}_2 = F\mathbf{A}_1 + \overline{F}\mathbf{A}_2, \tag{6·18}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are the covariant base vectors in the complex co-ordinate system $(z, \overline{z})$. Then, remembering (6·5), we may interpret (4·8) in complex co-ordinates to get

$$F = 2\mathrm{i}\frac{\partial\phi}{\partial \overline{z}}. \tag{6·19}$$

Similarly, for the couple M about the origin we obtain from (4·11)

$$M = z\frac{\partial\phi}{\partial z} + \overline{z}\frac{\partial\phi}{\partial \overline{z}} - \phi. \tag{6·20}$$

From (6·19), or directly from (4·12), at all points of a boundary curve which is entirely free from applied stress, we have

$$\frac{\partial\phi}{\partial z} = 0, \tag{6·21}$$

together with the complex conjugate of this equation.

By introducing (6·16) into (6·17), (6·19), (6·20) and (6·21) we may readily obtain the corresponding relations for complex co-ordinates $(\zeta, \overline{\zeta})$ in the undeformed body, but these are not required for subsequent applications.

7. Successive approximations: incompressible materials

The classical infinitesimal theory of plane stress is obtained by neglecting squares and products of the displacement D and its derivatives with respect to z, $\overline{z}$ or ζ, $\overline{\zeta}$ in the equations of the previous section. Further approximations based on the classical theory may be obtained by considering higher order terms in these relations. Taking co-ordinates $(z, \overline{z})$ in the deformed body we put

$$D = \epsilon\{{}^0D(z,\overline{z})\} + \epsilon^2\{{}^1D(z,\overline{z})\} + \ldots, \tag{7·1}$$

where ϵ is a characteristic real parameter in a given problem. Also, since λ is the ratio of the thickness of the plate after deformation to that before deformation we may write

$$\lambda = 1 + \epsilon\{{}^0\lambda(z,\overline{z})\} + \epsilon^2\{{}^1\lambda(z,\overline{z})\} + \ldots. \tag{7·2}$$

For incompressible materials we thus obtain from (6·11)

$$\left.\begin{aligned} I_1 &= 3 + \epsilon^2\left\{3({}^0\lambda)^2 + 4\frac{\partial\,{}^0D}{\partial \overline{z}}\frac{\partial\,{}^0\overline{D}}{\partial z}\right\} + \ldots, \\ I_2 &= 3 + \epsilon^2\left\{3({}^0\lambda)^2 + 4\frac{\partial\,{}^0D}{\partial \overline{z}}\frac{\partial\,{}^0\overline{D}}{\partial z}\right\} + \ldots. \end{aligned}\right\} \tag{7·3}$$

In the present paper we shall confine our attention to terms of the first and second orders in ϵ in the expansion (7·1) for D, and to this degree of approximation the form of strain-energy function suggested by Mooney (1940) is adequate. We may thus write

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{7·4}$$

so that C_1, C_2 are the values of $\partial W/\partial I_1$, $\partial W/\partial I_2$ respectively at $I_1 = I_2 = 3$. Also we may put

$$\phi = {}^0H\epsilon\{{}^0\phi(z,\bar{z})+\epsilon\,{}^1\phi(z,\bar{z})+\ldots\}, \tag{7·5}$$

where ${}^0H$ is a constant, which, for convenience, we shall choose to have the value $4h_0(C_1+C_2)$ so that, from (5·14), $H = {}^0H$ when $\epsilon = 0$.

Introducing the relations (7·1) to (7·5) into (6·10) and (6·15), and equating to zero the coefficients of ϵ in the resulting equations, we obtain

$$\left.\begin{aligned}\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+{}^0\lambda &= 0,\\ \frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\,{}^0\bar{D}}{\partial z} &= 0,\\ \frac{\partial^2({}^0\phi)}{\partial z\,\partial\bar{z}}+\frac{3\;{}^0\lambda}{2} &= 0.\end{aligned}\right\} \tag{7·6}$$

Similarly the coefficients of ϵ^2 in these equations yield

$$\left.\begin{aligned}\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}+{}^1\lambda &= \frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}-\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z},\\ \frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\bar{D}}{\partial z} &= \frac{\partial\,{}^0\bar{D}}{\partial z}\left(\frac{\partial\,{}^0D}{\partial z}+\alpha\,{}^0\lambda\right),\\ \frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}+\frac{3\,{}^1\lambda}{2} &= \alpha\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}-\tfrac{3}{4}(1+\alpha)\,({}^0\lambda)^2,\end{aligned}\right\} \tag{7·7}$$

where $\alpha = (C_1-C_2)/(C_1+C_2)$. Similar equations may be obtained from the coefficients of higher powers of ϵ provided higher order terms than those given in (7·4) in the expansion for the strain-energy function W are taken into account. The first approximation corresponds to the classical theory, and the equations for this may be integrated in terms of complex potential functions $\Omega(z)$, $\omega(z)$. Thus, from (7·6),

$$\left.\begin{aligned}{}^0\phi(z,\bar{z}) &= \bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ {}^0D(z,\bar{z}) &= \tfrac{5}{3}\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}),\\ {}^0\lambda(z,\bar{z}) &= -\tfrac{2}{3}\{\Omega'(z)+\bar{\Omega}'(\bar{z})\},\end{aligned}\right\} \tag{7·8}$$

a prime indicating the derivative of a function with respect to its argument.

Eliminating ${}^1\lambda$ between the first and third of equations (7·7) we obtain

$$\begin{aligned}\frac{2}{3}\frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}-\frac{\partial\,{}^1D}{\partial z}-\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}} &= \tfrac{1}{3}(2\alpha+3)\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}-\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}-\tfrac{1}{2}(1+\alpha)\,({}^0\lambda)^2\\ &= \tfrac{1}{3}(2\alpha+3)\,\{\bar{z}\Omega''(z)+\omega''(z)\}\,\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\\ &\quad-\tfrac{1}{9}(2\alpha-13)\,\{[\Omega'(z)]^2+[\bar{\Omega}'(\bar{z})]^2\}-\tfrac{2}{9}(2\alpha+19)\,\Omega'(z)\,\bar{\Omega}'(\bar{z}),\end{aligned} \tag{7·9}$$

if we make use of (7·8). Similarly, again using (7·8), the second of equations (7·7) becomes

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\, {}^1\overline{D}}{\partial z}=\tfrac{1}{3}\{\bar{z}\Omega''(z)+\omega''(z)\}\{(2\alpha-5)\,\Omega'(z)+(2\alpha+3)\,\overline{\Omega}'(\bar{z})\}. \qquad (7{\cdot}10)$$

This equation may be integrated to yield

$$\frac{\partial^1\phi}{\partial z}+{}^1\overline{D}=\tfrac{8}{3}\overline{\Delta}(\bar{z})+\tfrac{1}{3}(2\alpha-5)\left\{\tfrac{1}{2}\bar{z}[\Omega'(z)]^2+\int^z\Omega'(z)\,\omega''(z)\,dz\right\}$$
$$-\tfrac{1}{18}(2\alpha+11)\int^{\bar{z}}\{\overline{\Omega}'(\bar{z})\}^2\,d\bar{z}+\tfrac{1}{3}(2\alpha+3)\,\overline{\Omega}'(\bar{z})\,\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}, \qquad (7{\cdot}11)$$

where $\overline{\Delta}(\bar{z})$ is an arbitrary function of $\bar{z}$ and the additional terms in $\bar{z}$ have been inserted to simplify subsequent expressions. From (7·9) and (7·11) it follows that

$$\frac{8}{3}\frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}=\tfrac{8}{3}\{\Delta'(z)+\overline{\Delta}'(\bar{z})\}+\tfrac{4}{9}(2\alpha-5)\,\Omega'(z)\,\overline{\Omega}'(\bar{z})$$
$$+\tfrac{1}{3}(2\alpha+3)\,\{\Omega''(z)\,[z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)]+\overline{\Omega}''(\bar{z})\,[\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})]$$
$$+[\bar{z}\Omega''(z)+\omega''(z)]\,[z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})]-\tfrac{5}{3}[\Omega'(z)]^2-\tfrac{5}{3}[\overline{\Omega}'(\bar{z})]^2\},$$

and hence, by integration,

$$\frac{\partial\, {}^1\phi(z,\bar{z})}{\partial\bar{z}}=\Delta(z)+z\overline{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+\tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\overline{\Omega}'(\bar{z})$$
$$+(2\alpha+3)\,\{\tfrac{1}{8}\Gamma_1(z,\bar{z})-\tfrac{1}{3}z[\overline{\Omega}'(\bar{z})]^2\}, \qquad (7{\cdot}12)$$

where

$$\Gamma_1(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}$$
$$+\{\Omega'(z)+\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)\}$$
$$=-\left\{{}^0D\frac{\partial}{\partial z}+{}^0\overline{D}\frac{\partial}{\partial\bar{z}}\right\}\frac{\partial\, {}^0\phi}{\partial\bar{z}}, \qquad (7{\cdot}13)$$

and $\bar{\delta}'(\bar{z})$ is a further arbitrary function of $\bar{z}$. By integration of (7·12) we may obtain an expression for ${}^1\phi$, but this is not required in applications of the theory. From (7·11) and (7·12)

$${}^1D(z,\bar{z})=\tfrac{5}{3}\Delta(z)-z\overline{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-\tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\overline{\Omega}'(\bar{z})$$
$$-\tfrac{1}{8}(2\alpha+3)\,\Lambda_1(z,\bar{z})+\tfrac{1}{6}(6\alpha+1)\,z\{\overline{\Omega}'(\bar{z})\}^2$$
$$-\tfrac{1}{18}(2\alpha+11)\int^z[\Omega'(z)]^2\,dz+\tfrac{1}{3}(2\alpha-5)\int^{\bar{z}}\overline{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,d\bar{z}, \qquad (7{\cdot}14)$$

where

$$\Lambda_1(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\tfrac{5}{3}\overline{\Omega}(\bar{z})\}$$
$$-\{\tfrac{5}{3}\Omega'(z)-\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\tfrac{5}{3}\Omega(z)\}$$
$$=\left({}^0D\frac{\partial}{\partial z}+{}^0\overline{D}\frac{\partial}{\partial\bar{z}}\right){}^0D. \qquad (7{\cdot}15)$$

An expression for ${}^1\lambda(z,\bar{z})$ in terms of complex potential functions may now be obtained, if required, by introducing (7·14) and (7·8) into the first of equations (7·7).

For problems which are non-dislocational in character the complex potential functions $\Omega(z)$, $\omega(z)$, $\Delta(z)$ and $\delta(z)$ must be chosen so that the stress and displacement functions are single-valued. It follows that ${}^0D$, ${}^1D, \ldots, {}^0\lambda$, ${}^1\lambda, \ldots$, and all their derivatives with respect to z and $\bar{z}$, and similarly the second and higher order derivatives of ${}^0\phi$, ${}^1\phi$ ..., if they exist, must be single-valued at interior points of the body. Thus from (7·8)

$$[\Omega'(z)]_C = 0, \quad [\omega''(z)]_C = 0, \quad 5[\Omega(z)]_C = 3[\bar{\omega}'(\bar{z})]_C, \tag{7·16}$$

and similarly from these relations, with (7·12) and (7·14), we may infer that

$$\left.\begin{aligned} &[\Delta'(z)]_C = 0, \quad [12\delta''(z) + (6\alpha-7)\,\bar{\Omega}(\bar{z})\,\Omega''(z)]_C = 0, \\ &[5\Delta(z) - 3\bar{\delta}'(\bar{z})]_C \\ &\quad = \left[\tfrac{1}{6}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z - (2\alpha-5)\int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z} + \tfrac{1}{4}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z})\right]_C, \end{aligned}\right\} \tag{7·17}$$

where in (7·16) and (7·17), $[\,]_C$ denotes the change in value of the function inside the brackets during a complete circuit of a contour C lying entirely within the deformed body.†

For some problems it is convenient to remove the integral terms from (7·14). Replacing $\Delta(z)$ by $\Delta(z) + \frac{1}{30}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z$ and $\delta'(z)$ by $\delta'(z) + \frac{1}{3}(2\alpha-5)\int^z \Omega'(z)\,\omega''(z)\,\mathrm{d}z$ in (7·12) and (7·14) we obtain

$$\left.\begin{aligned} \frac{\partial\, {}^1\phi(z,\bar{z})}{\partial\bar{z}} &= \Delta(z) + z\bar{\Delta}'(\bar{z}) + \bar{\delta}'(\bar{z}) + \tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z}) \\ &\quad + \tfrac{1}{8}(2\alpha+3)\,\Gamma_1(z,\bar{z}) - \tfrac{1}{30}(18\alpha+19)\,z\{\bar{\Omega}'(\bar{z})\}^2 \\ &\quad + \tfrac{1}{30}(2\alpha+11)\int^z \{\Omega'(z)\}^2\,\mathrm{d}z + \tfrac{1}{3}(2\alpha-5)\int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}, \\ {}^1D(z,\bar{z}) &= \tfrac{5}{3}\Delta(z) - z\bar{\Delta}'(\bar{z}) - \bar{\delta}'(\bar{z}) - \tfrac{1}{12}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z}) \\ &\quad + \tfrac{1}{15}(14\alpha-3)\,z\{\bar{\Omega}'(\bar{z})\}^2 - \tfrac{1}{8}(2\alpha+3)\,\Lambda_1(z,\bar{z}). \end{aligned}\right\} \tag{7·18}$$

The conditions (7·17) for single-valued stress resultants and displacements now, however, reduce to

$$\left.\begin{aligned} &[\Delta'(z)]_C = 0, \quad [12\delta''(z) + (6\alpha-7)\,\bar{\Omega}(\bar{z})\,\Omega''(z)]_C = 0, \\ &[5\Delta(z) - 3\delta'(\bar{z}) - \tfrac{1}{4}(6\alpha-7)\,\Omega(z)\,\bar{\Omega}'(\bar{z})]_C = 0. \end{aligned}\right\} \tag{7·19}$$

The corresponding results for complex co-ordinates $(\zeta, \bar{\zeta})$ in the undeformed body may readily be obtained by expanding the argument z in $\phi(z,\bar{z})$, $D(z,\bar{z})$ by means of (6·3). If we express D in the form

$$D = \epsilon\{{}^0D'(\zeta,\bar{\zeta})\} + \epsilon^2\{{}^1D'(\zeta,\bar{\zeta})\} + \dots, \tag{7·20}$$

and introduce this expansion, together with (6·3), into (7·1) we obtain

$$D = \epsilon\{{}^0D(\zeta,\bar{\zeta})\} + \epsilon^2\left\{{}^1D(\zeta,\bar{\zeta}) + {}^0D'(\zeta,\bar{\zeta})\,\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\zeta} + {}^0\bar{D}'(\zeta,\bar{\zeta})\,\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\bar{\zeta}}\right\} + \dots. \tag{7·21}$$

Comparing (7·20) and (7·21) and making use of (7·15) we thus have

$$\left.\begin{aligned} {}^0D'(\zeta,\bar{\zeta}) &= {}^0D(\zeta,\bar{\zeta}), \\ {}^1D'(\zeta,\bar{\zeta}) &= {}^1D(\zeta,\bar{\zeta}) + \Lambda_1(\zeta,\bar{\zeta}), \end{aligned}\right\} \tag{7·22}$$

† The conditions (10·22) given by Adkins *et al.* (1953) for plane strain, when the resultant force on the contour is zero, are, of course, only true if the integral terms in the preceding equation (10·16) are single-valued. This is the case for the examples considered in that paper.

and similarly from (7·5) and (7·13) we may obtain

$$\left.\begin{aligned} \frac{\partial\ {}^0\phi(z,\bar{z})}{\partial \bar{z}} &= \frac{\partial\ {}^0\phi(\zeta,\bar{\zeta})}{\partial \bar{\zeta}}, \\ \frac{\partial\ {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \frac{\partial\ {}^1\phi(\zeta,\bar{\zeta})}{\partial \bar{\zeta}} - \Gamma_1(\zeta,\bar{\zeta}), \end{aligned}\right\} \tag{7·23}$$

where $\Gamma_1(\zeta,\bar{\zeta})$, $\Lambda_1(\zeta,\bar{\zeta})$ are obtained by replacing $z, \bar{z}$ by $\zeta, \bar{\zeta}$ in (7·13) and (7·15) respectively. The first-order stress and displacement functions ${}^0\phi(\zeta,\bar{\zeta})$ and ${}^0D'(\zeta,\bar{\zeta})$ are thus obtained by replacing $z, \bar{z}$ by $\zeta, \bar{\zeta}$ in (7·8). Also, combining the second of equations (7·22) and (7·23) with (7·12) and (7·14) we obtain

$$\left.\begin{aligned} \frac{\partial\ {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \Delta(\zeta) + \zeta\bar{\Delta}'(\bar{\zeta}) + \bar{\delta}'(\bar{\zeta}) + \tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta}) \\ &\quad + \tfrac{1}{8}(2\alpha-5)\,\Gamma_1(\zeta,\bar{\zeta}) - \tfrac{1}{3}(2\alpha+3)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2, \\ {}^1D'(\zeta,\bar{\zeta}) &= \tfrac{5}{3}\Delta(\zeta) - \zeta\bar{\Delta}'(\bar{\zeta}) - \bar{\delta}'(\bar{\zeta}) - \tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta}) \\ &\quad + \tfrac{1}{6}(6\alpha+1)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2 - \tfrac{1}{18}(2\alpha+11)\int^{\zeta}\{\Omega'(\zeta)\}^2\,\mathrm{d}\zeta \\ &\quad - (2\alpha-5)\left\{\tfrac{1}{8}\Lambda_1(\zeta,\bar{\zeta}) - \tfrac{1}{3}\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\right\}, \end{aligned}\right\} \tag{7·24}$$

and similarly from (7·18) we have the alternative forms

$$\left.\begin{aligned} \frac{\partial\ {}^1\phi(z,\bar{z})}{\partial \bar{z}} &= \Delta(\zeta) + \zeta\bar{\Delta}'(\bar{\zeta}) + \bar{\delta}'(\bar{\zeta}) + \tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta}) \\ &\quad + \frac{1}{30}\left\{(2\alpha+11)\int^{\zeta}[\Omega'(\zeta)]^2\,\mathrm{d}\zeta - (18\alpha+19)\,\zeta[\bar{\Omega}'(\bar{\zeta})]^2\right\} \\ &\quad + (2\alpha-5)\left\{\tfrac{1}{8}\Gamma_1(\zeta,\bar{\zeta}) + \frac{1}{3}\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\right\}, \\ {}^1D'(\zeta,\bar{\zeta}) &= \tfrac{5}{3}\Delta(\zeta) - \zeta\bar{\Delta}'(\bar{\zeta}) - \bar{\delta}'(\bar{\zeta}) - \tfrac{1}{12}(6\alpha-7)\,\Omega(\zeta)\,\bar{\Omega}'(\bar{\zeta}) \\ &\quad + \tfrac{1}{15}(14\alpha-3)\,\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2 - \tfrac{1}{8}(2\alpha-5)\,\Lambda_1(\zeta,\bar{\zeta}). \end{aligned}\right\} \tag{7·25}$$

The conditions for (7·24) and (7·25) to yield single-valued stress resultant and displacement components now take the forms (7·17) and (7·19) respectively with $\zeta, \bar{\zeta}$ replacing $z, \bar{z}$.

Expressions for the stress components, and for the resultant force and couple acting on a curve in the deformed body may now be obtained in terms of complex potential functions by combining the expressions obtained in the present section for ${}^0\phi$, ${}^1\phi$, ${}^0D$ and ${}^1D$ with (7·1), (7·5), (6·17), (6·19) and (6·20).

8. Successive approximations: compressible materials

Approximate solutions of equations (6·13) and (6·14) for compressible materials may be obtained without difficulty in terms of complex potential functions by the methods of the previous section. Assuming expansions of the forms (7·1) and (7·2) for D and λ, we obtain from (6·7) and (6·9)

$$\left.\begin{aligned} J_1 &= \epsilon\ {}^0J_1 + \epsilon^2({}^1J_1) + \ldots, \\ J_2 &= \epsilon^2({}^1J_2) + \ldots, \\ J_3 &= O(\epsilon^3), \end{aligned}\right\} \tag{8·1}$$

where

$$\left.\begin{aligned} {}^0J_1 &= 2\left\{\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+{}^0\lambda\right\},\\ {}^1J_1 &= 2\left\{\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}+{}^1\lambda\right\}+2\left\{\left(\frac{\partial\,{}^0D}{\partial z}\right)^2+\left(\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2+\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+3\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}\right\}+({}^0\lambda)^2,\\ {}^1J_2 &= \left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2-4\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}+4{}^0\lambda\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right), \end{aligned}\right\} \quad (8{\cdot}2)$$

and

$$\begin{aligned}\frac{\sqrt{I_3}}{\lambda} = 1&+\epsilon\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)\\ &+\epsilon^2\left\{\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}+\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2-\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}\right\}+\ldots. \end{aligned} \quad (8{\cdot}3)$$

Also, since $W = W(J_1, J_2, J_3)$ we may obtain from (8·1)

$$\begin{aligned}\frac{\partial W}{\partial J_r} = \left[\frac{\partial W}{\partial J_r}\right]_0 &+\epsilon\,{}^0J_1\left[\frac{\partial^2 W}{\partial J_1\partial J_r}\right]_0\\ &+\epsilon^2\left\{{}^1J_1\left[\frac{\partial^2 W}{\partial J_1\partial J_r}\right]_0+{}^1J_2\left[\frac{\partial^2 W}{\partial J_2\partial J_r}\right]_0+\tfrac{1}{2}({}^0J_1)^2\left[\frac{\partial^3 W}{\partial J_1^2\partial J_r}\right]_0\right\}+\ldots \quad (r = 1, 2, 3),\end{aligned} \quad (8{\cdot}4)$$

where the suffix 0 indicates that the quantity inside the square brackets is evaluated at $J_1 = J_2 = J_3 = 0$. It has been pointed out by Murnaghan (1937) that for a material in which the stress is zero in the undeformed state, $[\partial W/\partial J_1]_0 = 0$, a result which may readily be obtained by considering the expansion of (6·14) in powers of ϵ. Also, by considering the first approximation terms in the stress-strain relations, it has been shown by Rivlin (1953) that the Lamé constants λ and μ of the classical theory of elasticity are given by

$$\lambda = 4\left\{\left[\frac{\partial W}{\partial J_2}\right]_0+\left[\frac{\partial^2 W}{\partial J_1^2}\right]_0\right\},\quad \mu = -2\left[\frac{\partial W}{\partial J_2}\right]_0. \quad (8{\cdot}5)$$

The Lamé constant λ in (8·5) is not used again so it need not be confused with λ used elsewhere in the paper.

By analogy with §7 we may express ϕ in the form (7·5) where ${}^0H$ now has the value $-4h_0[\partial W/\partial J_2]_0$ so that, from (5·19) we again have $H = {}^0H$ when $\epsilon = 0$. Introducing (7·1), (7·2), (7·5) and (8·1) to (8·4) into (6·13) and (6·14) and equating the coefficients of ϵ in the resulting equations to zero we obtain

$$\left.\begin{aligned} &\frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\,{}^0\bar{D}}{\partial z} = 0,\\ &2\frac{\partial^2({}^0\phi)}{\partial z\partial \bar{z}}+(2c_1+1)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)+2(c_1+1){}^0\lambda = 0,\\ &(c_1+1)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)+c_1\,{}^0\lambda = 0, \end{aligned}\right\} \quad (8{\cdot}6)$$

and similarly from the coefficients of ϵ^2 we have

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\bar{D}}{\partial z} = \frac{\partial\,{}^0\bar{D}}{\partial z}\left\{2(c_1-c_2)\frac{\partial\,{}^0D}{\partial z}+(2c_1-2c_2-1)\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+2(c_1-c_2-c_3+1)\,{}^0\lambda\right\}, \quad (8{\cdot}7a)$$

$$2\frac{\partial^2(^1\phi)}{\partial z\partial\bar{z}}+(2c_1+1)\left(\frac{\partial\,^1D}{\partial z}+\frac{\partial\,^1\bar{D}}{\partial\bar{z}}\right)+2(c_1+1)\,^1\lambda$$
$$+(2c_1+3c_2+2c_4+1)\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)^2-(2c_1+1)\frac{\partial\,^0D}{\partial z}\frac{\partial\,^0\bar{D}}{\partial\bar{z}}+(6c_1-4c_2-1)\frac{\partial\,^0\bar{D}}{\partial z}\frac{\partial\,^0D}{\partial\bar{z}}$$
$$+2(5c_2+c_3+2c_4)\,^0\lambda\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)+(c_1+4c_2+2c_4+1)\,(^0\lambda)^2=0, \tag{8·7b}$$

$$2(c_1+1)\left(\frac{\partial\,^1D}{\partial z}+\frac{\partial\,^1\bar{D}}{\partial\bar{z}}\right)+2c_1\,^1\lambda+(2c_1+5c_2+c_3+2c_4+2)\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)^2$$
$$-2(c_1+1)\frac{\partial\,^0D}{\partial z}\frac{\partial\,^0\bar{D}}{\partial\bar{z}}+2(3c_1-2c_2-2c_3+3)\frac{\partial\,^0D}{\partial\bar{z}}\frac{\partial\,^0\bar{D}}{\partial z}$$
$$+4(2c_2+c_4)\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)\,^0\lambda+(c_1+2c_4)\,(^0\lambda)^2=0, \tag{8·7c}$$

where we have written

$$\left.\begin{aligned}\left[\frac{\partial^2W}{\partial J_1^2}\right]_0\Big/\left[\frac{\partial W}{\partial J_2}\right]_0&=c_1,\quad \left[\frac{\partial^2W}{\partial J_1\partial J_2}\right]_0\Big/\left[\frac{\partial W}{\partial J_2}\right]_0=c_2,\\ \left[\frac{\partial W}{\partial J_3}\right]_0\Big/\left[\frac{\partial W}{\partial J_2}\right]_0&=c_3,\quad \left[\frac{\partial^3W}{\partial J_1^3}\right]_0\Big/\left[\frac{\partial W}{\partial J_2}\right]_0=c_4.\end{aligned}\right\} \tag{8·8}$$

Equations (8·6) may be solved in terms of complex potential functions $\Omega(z)$, $\omega(z)$ to yield

$$\left.\begin{aligned}^0\phi(z,\bar{z})&=\bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ ^0D(z,\bar{z})&=\kappa\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}),\\ ^0\lambda(z,\bar{z})&=\tfrac{1}{2}(\kappa-3)\,\{\Omega'(z)+\bar{\Omega}'(\bar{z})\},\end{aligned}\right\} \tag{8·9}$$

where $$\kappa=\frac{5c_1+2}{3c_1+2},\quad \kappa-3=-\frac{4(c_1+1)}{3c_1+2}. \tag{8·10}$$

We may observe that the constant c_1 can be expressed in terms of Poisson's ratio η by the relation $c_1=-(1-\eta)/(1-2\eta)$ so that $\kappa=(3-\eta)/(1+\eta)$ and $\kappa-3=-4\eta/(1+\eta)$. Since, from (8·5), the modulus of rigidity μ of the material is contained in the constant ^0H, equations (8·9) are equivalent to the usual formulae of the classical theory of generalized plane stress.

Introducing the expressions (8·9) into (8·7*a*), and remembering (8·10), we obtain

$$\frac{\partial^2(^1\phi)}{\partial z^2}+\frac{\partial\,^1\bar{D}}{\partial z}=\{\bar{z}\Omega''(z)+\omega''(z)\}\{B_1'\Omega'(z)+B_1\bar{\Omega}'(\bar{z})\}, \tag{8·11}$$

where we have written for brevity

$$\left.\begin{aligned}B_1&=\{13c_1+6-4c_2-4(c_1+1)\,c_3\}/(3c_1+2),\\ B_1'&=\{5c_1+2-4c_2-4(c_1+1)\,c_3\}/(3c_1+2).\end{aligned}\right\} \tag{8·12}$$

Similarly, by eliminating $^1\lambda$ between equations (8·7*b*) and (8·7*c*) we have

$$2c_1\frac{\partial^2(^1\phi)}{\partial z\partial\bar{z}}-(3c_1+2)\left(\frac{\partial\,^1D}{\partial z}+\frac{\partial\,^1\bar{D}}{\partial\bar{z}}\right)$$
$$=\{3c_1+2+(2c_1+5)\,c_2+(c_1+1)\,c_3+2c_4\}\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)^2$$
$$-(3c_1+2)\frac{\partial\,^0D}{\partial z}\frac{\partial\,^0\bar{D}}{\partial\bar{z}}+\{13c_1+6-4c_2-4(c_1+1)\,c_3\}\frac{\partial\,^0D}{\partial\bar{z}}\frac{\partial\,^0\bar{D}}{\partial z}$$
$$-2\{(c_1-4)\,c_2+c_1c_3-2c_4\}\left(\frac{\partial\,^0D}{\partial z}+\frac{\partial\,^0\bar{D}}{\partial\bar{z}}\right)\,^0\lambda-2(2c_1c_2-c_4)\,(^0\lambda)^2, \tag{8·13}$$

or, making use of (8·9), (8·10) and (8·12)

$$(\kappa-1)\frac{\partial^2({}^1\phi)}{\partial z\,\partial\bar{z}}-\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial\bar{z}}\right)$$
$$=B_1\{\bar{z}\Omega''(z)+\omega''(z)\}\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}-B_2'\{[\Omega'(z)]^2+[\bar{\Omega}'(\bar{z})]^2\}-2B_2\,\Omega'(z)\,\bar{\Omega}'(\bar{z}),\qquad(8\cdot14)$$

where

$$\left.\begin{aligned}B_2&=\{(3c_1+2)\,(13c_1^2+16c_1+4)+4[3c_1(3c_1+4)\,c_2-3c_1^2(c_1+1)\,c_3-2c_4]\}/(3c_1+2)^3,\\B_2'&=-\{(3c_1+2)\,(19c_1^2+16c_1+4)-4[3c_1(3c_1+4)\,c_2-3c_1^2(c_1+1)\,c_3-2c_4]\}/(3c_1+2)^3.\end{aligned}\right\}\qquad(8\cdot15)$$

It will be observed that (8·11) and (8·14) are similar in form to the corresponding equations (7·10) and (7·9) for an incompressible material, and an analogous procedure may therefore be adopted to obtain expressions for $\partial^1\phi/\partial\bar{z}$ and ${}^1D$ in terms of complex potential functions. Thus

$$\left.\begin{aligned}\frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}}&=\Delta(z)+z\bar{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+B_3\Omega(z)\,\bar{\Omega}'(\bar{z})+\frac{B_1}{\kappa+1}\Gamma_2(z,\bar{z})-B_1z\{\bar{\Omega}'(\bar{z})\}^2,\\ \text{and}\qquad {}^1D(z,\bar{z})&=\kappa\Delta(z)-z\bar{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-B_3\,\Omega(z)\,\bar{\Omega}'(\bar{z})-\frac{B_1}{\kappa+1}\Lambda_2(z,\bar{z})\\&\quad+B_1''z\{\bar{\Omega}'(\bar{z})\}^2+B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}-B_4\int^{z}\{\Omega'(z)\}^2\,\mathrm{d}z,\end{aligned}\right\}\qquad(8\cdot16)$$

where

$$\left.\begin{aligned}B_1''&=B_1+\tfrac{1}{2}B_1'=\tfrac{1}{2}\{31c_1+14-12[c_2+(c_1+1)\,c_3]\}/(3c_1+2),\\B_3&=B_1-2B_2/(\kappa+1)\\&=\{(3c_1+2)\,(39c_1^2+34c_1+8)-4c_2(21c_1^2+26c_1+4)\\&\qquad-4c_3(c_1+1)\,(9c_1^2+14c_1+4)+8c_4\}/\{2(2c_1+1)\,(3c_1+2)^2\},\\B_4&=\tfrac{1}{2}B_1'-B_2'\\&=\{(3c_1+2)\,(53c_1^2+48c_1+12)-4c_2(27c_1^2+36c_1+4)\\&\qquad-4c_3(c_1+1)\,(3c_1^2+12c_1+4)+16c_4\}/\{2(3c_1+2)^3\},\end{aligned}\right\}\qquad(8\cdot17)$$

$$\left.\begin{aligned}\Gamma_2(z,\bar{z})&=\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\kappa\bar{\Omega}(\bar{z})\}\\&\quad+\{\Omega'(z)+\bar{\Omega}'(\bar{z})\}\{z\bar{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\kappa\Omega(z)\}\\&=-\left\{{}^0D\frac{\partial}{\partial z}+{}^0\bar{D}\frac{\partial}{\partial\bar{z}}\right\}\frac{\partial\,{}^0\phi}{\partial\bar{z}},\\ \text{and}\qquad\Lambda_2(z,\bar{z})&=\{z\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\kappa\bar{\Omega}(\bar{z})\}\\&\quad-\{\kappa\Omega'(z)-\bar{\Omega}'(\bar{z})\}\{z\bar{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})-\kappa\Omega(z)\}\\&=\left\{{}^0D\frac{\partial}{\partial z}+{}^0\bar{D}\frac{\partial}{\partial\bar{z}}\right\}{}^0D.\end{aligned}\right\}\qquad(8\cdot18)$$

An expression for ${}^1\lambda(z,\bar{z})$ in terms of complex potential functions may now be obtained from (8·7), (8·9) and (8·16). From (8·9) and (8·16) the conditions for single-valued stress resultant and displacement components become

$$[\Omega'(z)]_C=0,\quad[\omega''(z)]_C=0,\quad[\kappa\Omega(z)-\bar{\omega}'(\bar{z})]_C=0,\qquad(8\cdot19)$$

$$\left.\begin{aligned}&[\Delta'(z)]_C=0,\quad[\delta''(z)+B_3\bar{\Omega}(\bar{z})\,\Omega''(z)]_C=0,\\&[\kappa\Delta(z)-\bar{\delta}'(\bar{z})]_C=\left[B_4\int^{z}\{\Omega'(z)\}^2\,\mathrm{d}z-B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,\mathrm{d}\bar{z}+B_3\,\Omega(z)\,\bar{\Omega}'(\bar{z})\right]_C.\end{aligned}\right\}\qquad(8\cdot20)$$

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The integral terms may, if required, be removed from the second of equations (8·16) by a process similar to that employed in §7. Thus, replacing $\Delta(z)$ by $\Delta(z)+(B_4/\kappa)\int^z\{\Omega'(z)\}^2\,dz$ and $\delta'(z)$ by $\delta'(z)+B_1'\int^z\Omega'(z)\,\omega''(z)\,dz$ we obtain

$$\left.\begin{aligned}\frac{\partial\,{}^1\phi(z,\bar z)}{\partial\bar z}&=\Delta(z)+z\bar\Delta'(\bar z)+\bar\delta'(\bar z)+B_3\Omega(z)\,\bar\Omega'(\bar z)\\&\quad+\frac{B_1}{\kappa+1}\Gamma_2(z,\bar z)-B_5z\{\bar\Omega'(\bar z)\}^2+B_1'\int^{\bar z}\bar\Omega'(\bar z)\,\bar\omega''(\bar z)\,d\bar z+\frac{B_4}{\kappa}\int^z\{\Omega'(z)\}^2\,dz,\\{}^1D(z,\bar z)&=\kappa\Delta(z)-z\bar\Delta'(\bar z)-\bar\delta'(\bar z)-B_3\Omega(z)\,\bar\Omega'(\bar z)-\frac{B_1}{\kappa+1}\Lambda_2(z,\bar z)+B_6z\{\bar\Omega'(\bar z)\}^2,\end{aligned}\right\}\quad(8\cdot21)$$

where

$$\left.\begin{aligned}B_5&=B_1-B_4/\kappa\\&=\{(3c_1+2)\,(7c_1+2)\,(11c_1+6)-4c_2(3c_1^2-4c_1+4)\\&\qquad-4c_3(c_1+1)\,(27c_1^2+20c_1+4)-16c_4\}/\{2(5c_1+2)\,(3c_1+2)^2\},\\B_6&=B_1''-B_4/\kappa\\&=\{(3c_1+2)\,(51c_1^2+42c_1+8)-4c_2(9c_1^2+6c_1+4)\\&\qquad-4c_3(c_1+1)\,(21c_1^2+18c_1+4)-8c_4\}/\{(5c_1+2)\,(3c_1+2)^2\}.\end{aligned}\right\}\quad(8\cdot22)$$

The conditions (8·20) for single-valued stress resultants and displacements now, however, reduce to

$$\left.\begin{aligned}&[\Delta'(z)]_C=0,\quad[\delta''(z)+B_3\bar\Omega(\bar z)\,\Omega''(z)]_C=0,\\&[\kappa\Delta(z)-\bar\delta'(\bar z)]_C=B_3[\Omega(z)\,\bar\Omega'(\bar z)]_C.\end{aligned}\right\}\quad(8\cdot23)$$

To obtain the corresponding results for complex co-ordinates $(\zeta,\bar\zeta)$ in the undeformed body we may again assume an expansion of the form (7·20) for D, and the formulae (7·22), (7·23) may then be applied with $\Gamma_2(\zeta,\bar\zeta)$, $\Lambda_2(\zeta,\bar\zeta)$ replacing $\Gamma_1(\zeta,\bar\zeta)$, $\Lambda_1(\zeta,\bar\zeta)$ respectively. The first approximation stress and displacement functions are thus given by (8·9) with $\zeta,\bar\zeta$ replacing $z,\bar z$, and for the second-order terms we have from (8·16)

$$\left.\begin{aligned}\frac{\partial\,{}^1\phi(z,\bar z)}{\partial\bar z}&=\Delta(\zeta)+\zeta\bar\Delta'(\bar\zeta)+\bar\delta'(\bar\zeta)+B_3\Omega(\zeta)\,\bar\Omega'(\bar\zeta)+\frac{B_1'}{\kappa+1}\Gamma_2(\zeta,\bar\zeta)-B_1\,\zeta\{\bar\Omega'(\bar\zeta)\}^2,\\{}^1D'(\zeta,\bar\zeta)&=\kappa\Delta(\zeta)-\zeta\bar\Delta'(\bar\zeta)-\bar\delta'(\bar\zeta)-B_3\Omega(\zeta)\,\bar\Omega'(\bar\zeta)\\&\quad-\frac{B_1'}{\kappa+1}\Lambda_2(\zeta,\bar\zeta)+B_1''\zeta\{\bar\Omega'(\bar\zeta)\}^2+B_1'\int^{\bar\zeta}\bar\Omega'(\bar\zeta)\,\bar\omega''(\bar\zeta)\,d\bar\zeta-B_4\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta.\end{aligned}\right\}\quad(8\cdot24)$$

Alternative expressions for $\partial^1\phi(z,\bar z)/\partial\bar z$ and ${}^1D'(\zeta,\bar\zeta)$ may be obtained from (8·21). The conditions for single-valued stress resultants and displacements are again given by (8·19), (8·20) and (8·23) with ζ, $\bar\zeta$ replacing z, $\bar z$. The stress resultants and the resultant force and couple across a curve in the deformed body may now be obtained in terms of complex potential functions by combining the expressions obtained for ϕ and D with (6·17), (6·19) and (6·20).

By considering the uniform dilatation of a compressible material under a finite pressure, Rivlin (1953) has shown that an incompressible material may be regarded as the limiting case of a compressible material obtained by letting $[\partial^2W/\partial J_1^2]_0$ and $[\partial^2W/\partial J_1\partial J_2]_0$ tend to infinity in such a manner that their difference remains finite. Comparison of the form of W for a compressible material with that for an incompressible material as far as the terms of the third order of smallness then yields

$$\left[\frac{\partial W}{\partial J_2}\right]_0 = -(C_1+C_2), \quad \left[\frac{\partial W}{\partial J_3}\right]_0 = -(C_1+2C_2), \tag{8·25}$$

where C_1, C_2 are the Mooney constants defined for equation (7·4). Thus, in the results of the present section the passage to the incompressible case may be achieved by inserting the conditions

$$\left.\begin{aligned} &c_1 \to \infty, \quad c_2 \to \infty, \quad c_1/c_2 \to 1, \\ &c_3 = \frac{C_1+2C_2}{C_1+C_2} = \tfrac{1}{2}(3-\alpha). \end{aligned}\right\} \tag{8·26}$$

For example, by introducing (8·26) into (8·16) we obtain (7·12) and (7·14), and the alternative equations for an incompressible material may be obtained from the corresponding relations for a compressible material in a similar manner.

PLANE STRAIN

9. Approximate theory for compressible materials

In the theory of finite plane strain developed by Adkins *et al.* (1953), the application of approximation methods was confined to deformations of incompressible materials. In the present section, the corresponding results will be obtained for compressible materials prior to a general formulation of the second-order theory of elasticity for two-dimensional problems.

Employing the notation of § 2, we suppose the elastic body to be deformed by a uniform finite extension parallel to the x_3-axis with constant extension ratio λ_0, and that subsequently the body receives a finite plane strain parallel to the (x_1, x_2) plane. Thus if we choose the moving curvilinear co-ordinate θ_3 so that $\theta_3 = y_3$ then

$$\left.\begin{aligned} &x_3 = y_3/\lambda_0 = \theta_3/\lambda_0, \\ \text{and} \qquad &x_\alpha = x_\alpha(\theta_1, \theta_2), \quad y_\alpha = y_\alpha(\theta_1, \theta_2, t). \end{aligned}\right\} \tag{9·1}$$

Comparing (9·1) with (3·1) and (5·1), we see that equations (3·2) and (3·3) again apply, and that the analysis given in § 5 for compressible materials in plane stress may be repeated, with appropriate modifications, in the present instance. Thus, equations (5·2) to (5·6) are now satisfied exactly, provided we replace λ by λ_0 throughout. Also, from (5·6), (2·5), (5·15) and (5·16) we obtain

$$\left.\begin{aligned} \tau^{\alpha\beta} &= \frac{2}{\sqrt{I_3}}\left\{\frac{\partial W}{\partial J_1} + (\lambda_0^2-2)\frac{\partial W}{\partial J_2} - (\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}a^{\alpha\beta} + \frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_2} + (\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}A^{\alpha\beta}, \\ \tau^{33} &= \frac{2}{\sqrt{I_3}}\left\{\lambda_0^2\frac{\partial W}{\partial J_1} + \lambda_0^2(J_1+1-\lambda_0^2)\frac{\partial W}{\partial J_2} + [(1-\lambda_0^2)(J_1+1-\lambda_0^2)+J_2+J_3]\frac{\partial W}{\partial J_3}\right\}. \end{aligned}\right\} \tag{9·2}$$

The stress components may be expressed in terms of an Airy stress function ϕ by relations analogous to (4·5) and (4·6) so that we may write

$$\phi\,\|_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\rho}\tau^{\gamma\rho} = (A/a)\,({}_0\epsilon_{\alpha\gamma})\,({}_0\epsilon_{\beta\rho})\,\tau^{\gamma\rho}, \tag{9·3}$$

and the relations (4·8) to (4·12) may also be employed if **F** and **M** now denote the force and couple respectively across the arc AP of the plane $y_3 = 0$ in the deformed body, measured per unit length of the y_3-axis.

The complex co-ordinate systems $(\zeta, \bar{\zeta})$ and $(z, \bar{z})$ may now be defined as in §6 and it is at once evident that the relations (6·1) to (6·9) and (6·18) to (6·21) are again satisfied, with λ replaced throughout by λ_0. Moreover the equations of equilibrium may be expressed in forms analogous to (6·13). Thus we have

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= \frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_1}+(\lambda_0^2-2)\frac{\partial W}{\partial J_2}-(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial\bar{z}} &= \frac{1}{\lambda_0}\left\{\frac{\partial W}{\partial J_1}+\left(\frac{\sqrt{I_3}}{\lambda_0}+\lambda_0^2-2\right)\frac{\partial W}{\partial J_2}+(\lambda_0^2-1)\left(\frac{\sqrt{I_3}}{\lambda_0}-1\right)\frac{\partial W}{\partial J_3}\right\}\\ &\quad+\frac{2\sqrt{I_3}}{\lambda_0^2}\left\{\frac{\partial W}{\partial J_1}+(\lambda_0^2-2)\frac{\partial W}{\partial J_2}-(\lambda_0^2-1)\frac{\partial W}{\partial J_3}\right\}\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z},\end{aligned}\right\} \tag{9·4}$$

and these equations are sufficient for the determination of ϕ, D and $\bar{D}$. In applying approximation methods we shall confine our attention to plane strain for which $\lambda_0 = 1$, and equations (9·4) then reduce to

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial z^2} &= 2\sqrt{(I_3)}\left(\frac{\partial W}{\partial J_1}-\frac{\partial W}{\partial J_2}\right)\frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z}-1\right),\\ \frac{\partial^2\phi}{\partial z\partial\bar{z}} &= \frac{\partial W}{\partial J_1}+(\sqrt{I_3}-1)\frac{\partial W}{\partial J_2}+2\sqrt{(I_3)}\left(\frac{\partial W}{\partial J_1}-\frac{\partial W}{\partial J_2}\right)\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}.\end{aligned}\right\} \tag{9·5}$$

We now suppose the stress and displacement functions ϕ and D to be expanded in the forms (7·5) and (7·1), and we shall choose the constant ${}^0H$ to have the value $-2[\partial W/\partial J_2]_0$. From (8·25) we see that this choice is consistent with the value $2(C_1+C_2)$ employed by Adkins *et al.* in dealing with incompressible materials. From (6·9), with $\lambda = 1$, it is readily seen that $J_3 = 0$ and that the strain invariants J_1 and J_2 may be expanded in the forms (8·1), but now we have

$$\left.\begin{aligned}{}^0J_1 &= 2\left\{\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right\},\\ {}^1J_1 &= 2\left\{\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}+\left(\frac{\partial\,{}^0D}{\partial z}\right)^2+\left(\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2+\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+3\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}\right\},\\ {}^1J_2 &= \left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2-4\frac{\partial\,{}^0D}{\partial \bar{z}}\frac{\partial\,{}^0\bar{D}}{\partial z}.\end{aligned}\right\} \tag{9·6}$$

Also, (8·3) and (8·4) again apply with $\lambda = 1$. The relations for the determination of ${}^0\phi$, ${}^1\phi$, ${}^0D$ and ${}^1D$ may be obtained by a procedure analogous to that employed in §§7 and 8. This process yields

$$\left.\begin{aligned}\frac{\partial^2({}^0\phi)}{\partial z^2}+\frac{\partial\,{}^0\bar{D}}{\partial z} &= 0,\\ 2\frac{\partial^2({}^0\phi)}{\partial z\partial\bar{z}}+(2c_1+1)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right) &= 0,\end{aligned}\right\} \tag{9·7}$$

and

$$\left.\begin{aligned}&\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\bar{D}}{\partial z} = \frac{\partial\,{}^0\bar{D}}{\partial z}\left\{2(c_1-c_2)\frac{\partial\,{}^0D}{\partial z}+(2c_1-2c_2-1)\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right\},\\ &2\frac{\partial^2({}^1\phi)}{\partial z\partial\bar{z}}+(2c_1+1)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\bar{D}}{\partial \bar{z}}\right)+(2c_1+3c_2+2c_4+1)\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}\right)^2\\ &\qquad-(2c_1+1)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\bar{D}}{\partial \bar{z}}+(6c_1-4c_2-1)\frac{\partial\,{}^0\bar{D}}{\partial z}\frac{\partial\,{}^0D}{\partial \bar{z}} = 0.\end{aligned}\right\} \tag{9·8}$$

We may observe that these relations could have been obtained directly from the corresponding equations of (8·6) and (8·7) by putting $\lambda_0 = \lambda_1 = 0$. Moreover, since c_3 is absent from (9·8), we may infer from (8·25) that the stress and displacement functions for an incompressible material can only involve C_1 and C_2 in the form $(C_1 + C_2)$, a result obtained independently by Adkins *et al.*

Expressions for the stress and displacement functions in terms of the complex potential functions $\Omega(z)$, $\omega(z)$, $\Delta(z)$ and $\delta(z)$ may be obtained from (9·7) and (9·8) by a process similar to that employed for the corresponding equations in §§ 7 and 8. Thus from (9·7) we have

$$\left.\begin{aligned} {}^0\phi(z,\bar{z}) &= \bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}), \\ {}^0D(z,\bar{z}) &= \kappa\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}), \end{aligned}\right\} \tag{9·9}$$

where

$$\kappa = \frac{2c_1 - 1}{2c_1 + 1} = 3 - 4\eta. \tag{9·10}$$

This definition of κ is commonly used in the classical infinitesimal theory of plane strain. By combining (9·9) with (9·8) we may express the equations for the determination of ${}^1\phi$ and ${}^1D$ in the forms (8·11) and (8·14), in which κ now has the value (9·10), and the other constants are given by

$$\left.\begin{aligned} B_1 &= (6c_1 - 4c_2 - 1)/(2c_1 + 1), \\ B_1' &= (2c_1 - 4c_2 - 1)/(2c_1 + 1), \\ B_2 &= \{(2c_1 + 1)(4c_1^2 - 3) - 4(3c_2 + 2c_4)\}/(2c_1 + 1)^3, \\ B_2' &= -\{(2c_1 + 1)(4c_1^2 + 3) + 4(3c_2 + 2c_4)\}/(2c_1 + 1)^3. \end{aligned}\right\} \tag{9·11}$$

The solution may then be completed as in § 8 and the formulae there derived for the stress and displacement functions, from (8·16) onwards, now apply, provided we employ (9·10) and (9·11) to evaluate the remaining constants. Thus in (8·16), (8·19) to (8·21), (8·23) and (8·24) we now have

$$\left.\begin{aligned} B_1'' &= (14c_1 - 12c_2 - 3)/\{2(2c_1 + 1)\}, \\ B_3 &= \{(2c_1 + 1)(8c_1^2 - 2c_1 + 3) - 4c_2(4c_1^2 + 2c_1 - 3) + 8c_4\}/\{2c_1(2c_1 + 1)^2\}, \\ B_4 &= \{(2c_1 + 1)(12c_1^2 + 5) - 4c_2(4c_1^2 + 4c_1 - 5) + 16c_4\}/\{2(2c_1 + 1)^3\}, \\ B_5 &= \{(2c_1 + 1)(2c_1 - 3)(6c_1 + 1) - 4c_2(4c_1^2 - 4c_1 + 3) - 16c_4\}/\{2(2c_1 + 1)^2(2c_1 - 1)\}, \\ B_6 &= \{(2c_1 + 1)(8c_1^2 - 10c_1 - 1) - 4c_2(4c_1^2 - 2c_1 + 1) - 8c_4\}/\{(2c_1 + 1)^2(2c_1 - 1)\}, \end{aligned}\right\} \tag{9·12}$$

and in (8·18) the constant κ is given by (9·10).

The relations obtained by Adkins *et al.* (1953) for incompressible materials in plane strain may again be derived as limiting cases of these results by introduction of the conditions (8·26).

SECOND-ORDER THEORY FOR TWO-DIMENSIONAL PROBLEMS

10. General formulation

The similarity of the results obtained in §§ 7 to 9 suggests that the second-order theory for two-dimensional elasticity may be expressed in a more general form suitable for application to problems either in plane stress or plane strain. This has already been achieved to some extent for compressible materials in §§ 8 and 9. Thus the first two of equations (8·9), (8·11), (8·14), (8·16), (8·18) to (8·21), (8·23) and (8·24) apply for compressible materials in plane stress or plane strain, provided the appropriate values are chosen for the constant coefficients. For plane stress, these coefficients are given by (8·10), (8·12), (8·15), (8·17) and (8·22), while for plane strain the formulae (9·10) to (9·12) must be employed. Moreover, the passage to the incompressible case is effected in each instance by introducing the limiting conditions (8·26) into the formulae for the constants κ, B_1, B_1', etc., the equations for the determination of the stress and displacement functions remaining unchanged in form.

A further simplification may be achieved by writing equations (8·16) in the form

$$\left.\begin{aligned}\frac{\partial\, {}^1\phi(z,\bar{z})}{\partial\bar{z}} &= \Delta(z)+z\bar{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+B_3\Omega(z)\,\bar{\Omega}'(\bar{z})-B_1 z\{\bar{\Omega}'(\bar{z})\}^2+\gamma\Gamma(z,\bar{z}),\\ {}^1D(z,\bar{z}) &= \kappa\Delta(z)-z\bar{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-B_3\Omega(z)\,\bar{\Omega}'(\bar{z})+B_1'' z\{\bar{\Omega}'(\bar{z})\}^2+B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,d\bar{z}\\ &\quad -B_4\int^{z}\{\Omega'(z)\}^2\,dz-\gamma\Lambda(z,\bar{z}),\end{aligned}\right\} \tag{10·1}$$

where $\gamma = B_1/(\kappa+1)$ and for simplicity the suffixes have been dropped from Γ and Λ. From (8·24) it is now evident that the formulae (10·1) may be employed to obtain expressions for $\partial^1\phi(z,\bar{z})/\partial\bar{z}$ and ${}^1D'(\zeta,\bar{\zeta})$ in terms of complex co-ordinates $(\zeta,\bar{\zeta})$ in the undeformed body provided we replace z, $\bar{z}$ by ζ, $\bar{\zeta}$ throughout, and write $\gamma = B_1'/(\kappa+1)$. Similar formulae, with the integral terms removed from the expression for ${}^1D$, may be derived from (8·21). Moreover, from (7·22) and (7·23), the expressions for the first-order stress and displacement functions take the same forms in the co-ordinate systems $(z,\bar{z})$ and $(\zeta,\bar{\zeta})$. It follows, therefore, that we may employ equations (8·9) and (10·1), with the conditions (8·19) and (8·20), to determine values for the potential functions Ω, ω, Δ and δ satisfying a given set of boundary conditions, and from the results obtained, express the stress and displacement components in general forms which involve the constants κ, γ, B_1, B_1' etc. Alternatively, we may employ, in place of (10·1), the corresponding equations derived from (8·21), in which case the conditions (8·20) must be replaced by (8·23). From the general formulae thus derived, we may, by letting the constant coefficients take suitable values, deduce the corresponding results for plane stress or plane strain, with the given boundary conditions satisfied on a contour either in the deformed body or in the undeformed body. Since this procedure may be applied both to compressible materials and to incompressible materials, the single general solution may be made to yield, by insertion of the appropriate constants, the results for the eight associated problems shown in the following scheme:

$$\begin{Bmatrix}\text{plane stress}\\ \text{plane strain}\end{Bmatrix}\begin{Bmatrix}\text{compressible material}\\ \text{incompressible material}\end{Bmatrix}$$

$$\begin{Bmatrix}\text{boundary conditions specified on given contours in the undeformed body}\\ \text{boundary conditions specified on same contours in the deformed body}\end{Bmatrix}.$$

General expressions for the coefficients κ, γ, B_1, B_1', ... may be obtained from a further examination of the equations of §§ 8 and 9. For this purpose, we consider the general forms

$$\left.\begin{aligned}{}^0\phi(z,\bar{z}) &= \bar{z}\Omega(z)+z\bar{\Omega}(\bar{z})+\omega(z)+\bar{\omega}(\bar{z}),\\ {}^0D(z,\bar{z}) &= \frac{2k+3\beta}{2k+\beta}\,\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}),\\ {}^0\lambda(z,\bar{z}) &= -\frac{2(\beta+1)}{2k+\beta}\,\{\Omega'(z)+\bar{\Omega}'(\bar{z})\},\end{aligned}\right\} \tag{10·2}$$

$$\left.\begin{aligned}\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\overline{D}}{\partial z}&=\frac{\partial\,{}^0\overline{D}}{\partial z}\left\{2(k-c_2-1)\frac{\partial\,{}^0D}{\partial z}+(2k-2c_2-3)\frac{\partial\,{}^0\overline{D}}{\partial \bar z}+2(k-c_2-c_3)\,{}^0\lambda\right\},\\ 2\beta\frac{\partial^2({}^1\phi)}{\partial z\partial\bar z}-(2k+\beta)\left(\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\overline{D}}{\partial\bar z}\right)&=\{2k+\beta+(2\beta+5)\,c_2+(\beta+1)\,c_3+2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\overline{D}}{\partial\bar z}\right)^2\\ &\quad-(2k+\beta)\frac{\partial\,{}^0D}{\partial z}\frac{\partial\,{}^0\overline{D}}{\partial\bar z}+\{6k+7\beta-4c_2-4(\beta+1)\,c_3\}\frac{\partial\,{}^0D}{\partial\bar z}\frac{\partial\,{}^0\overline{D}}{\partial z}\\ &\quad-2\{(\beta-4)\,c_2+\beta c_3-2c_4\}\left(\frac{\partial\,{}^0D}{\partial z}+\frac{\partial\,{}^0\overline{D}}{\partial\bar z}\right){}^0\lambda-2(2\beta c_2-c_4)\,({}^0\lambda)^2,\end{aligned}\right\}\quad(10\cdot3)$$

where we have written $c_1+1=k$ to simplify the form of subsequent expressions. From (10·2) and (10·3) we may, by putting $\beta=k-1=c_1$ obtain (8·9), (8·7 *a*) and (8·13) which are appropriate to the case of plane stress, while the value $\beta=-1$ yields the corresponding equations (9·9) and (9·8) for plane strain. By introducing (10·2) into (10·3) and comparing the resulting equations with (8·11) and (8·14) we may thus express the constants B_1, B_1', B_2, B_2' in terms of the parameter β. To evaluate the remaining constants which occur in the expressions for the stress and displacement functions we may observe from (8·17) and (8·22) that

$$\left.\begin{aligned}&B_1''=B_1+\tfrac{1}{2}B_1',\quad B_3=B_1-2B_2/(\kappa+1),\quad B_4=\tfrac{1}{2}B_1'-B_2',\\ &B_5=B_1-B_4/\kappa,\quad B_6=B_1''-B_4/\kappa.\end{aligned}\right\}\quad(10\cdot4)$$

Remembering the definition of γ we thus obtain

$$\left.\begin{aligned}\kappa&=(2k+3\beta)/(2k+\beta),\\ B_1&=\{6k+7\beta-4c_2-4(\beta+1)\,c_3\}/(2k+\beta),\\ B_1'&=\{2k+3\beta-4c_2-4(\beta+1)\,c_3\}/(2k+\beta),\\ B_2&=\{(2k+\beta)\,[4(k+\beta)^2-3\beta^2]+12\beta(3\beta+4)\,c_2-12\beta^2(\beta+1)\,c_3-8c_4\}/(2k+\beta)^3,\\ B_2'&=-\{(2k+\beta)\,[4(k+\beta)^2+3\beta^2]-12\beta(3\beta+4)\,c_2+12\beta^2(\beta+1)\,c_3+8c_4\}/(2k+\beta)^3,\end{aligned}\right\}\quad(10\cdot5)$$

$$\left.\begin{aligned}B_1''&=\{14k+17\beta-12c_2-12(\beta+1)\,c_3\}/\{2(2k+\beta)\},\\ B_3&=\{(2k+\beta)\,(8k^2+18k\beta+13\beta^2)-4[4k^2+6(k+2)\,\beta+11\beta^2]\,c_2\\ &\qquad-4(\beta+1)\,(4k^2+6k\beta-\beta^2)\,c_3+8c_4\}/\{2(k+\beta)\,(2k+\beta)^2\},\\ B_4&=\{(2k+\beta)\,(12k^2+24k\beta+17\beta^2)-4[4k^2+4(k+6)\,\beta+19\beta^2]\,c_2\\ &\qquad-4(\beta+1)\,(4k^2+4k\beta-5\beta^2)\,c_3+16c_4\}/\{2(2k+\beta)^3\},\end{aligned}\right\}\quad(10\cdot6)$$

$$\left.\begin{aligned}B_5&=\{(2k+\beta)\,(2k+5\beta)\,(6k+5\beta)-4[4k^2+12(k-2)\,\beta-13\beta^2]\,c_2\\ &\qquad-4(\beta+1)\,(4k^2+12k\beta+11\beta^2)\,c_3-16c_4\}/\{2(2k+\beta)^2\,(2k+3\beta)\},\\ B_6&=\{(2k+\beta)\,(8k^2+26k\beta+17\beta^2)-4[4k^2+2(5k-6)\,\beta-5\beta^2]\,c_2\\ &\qquad-4(\beta+1)\,(4k^2+10k\beta+7\beta^2)\,c_3-8c_4\}/\{(2k+\beta)^2\,(2k+3\beta)\},\end{aligned}\right\}\quad(10\cdot7)$$

$$\gamma=\{2k+3\beta+4v(k+\beta)-4c_2-4(\beta+1)\,c_3\}/\{4(k+\beta)\},\qquad(10\cdot8)$$

where, in (10·8), $v=0$ for co-ordinates in the undeformed body and $v=1$ for co-ordinates in the deformed body. The constants for the incompressible case may be determined from (10·5) to (10·8) in any particular instance by proceeding to the limit, using (8·26), after the appropriate values of β and v have been inserted.

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Note added in proof (14 *October* 1954). The conditions (8·20) are simplified if, remembering (8·9), we replace $B_3\Omega(z)\bar{\Omega}'(\bar{z})$ by $B_3\bar{\Omega}'(\bar{z})\{{}^0D(z,\bar{z})+z\bar{\Omega}'(\bar{z})+\bar{\omega}'(\bar{z})\}/\kappa$ and $\bar{\delta}'(\bar{z})$ by $\bar{\delta}'(\bar{z})-(B_3/\kappa)\bar{\Omega}'(\bar{z})\bar{\omega}'(\bar{z})$. The terms $B_3\Omega(z)\bar{\Omega}'(\bar{z})-B_1z\{\bar{\Omega}'(\bar{z})\}^2$ and

$$-B_3\Omega(z)\bar{\Omega}'(\bar{z})+B_1''z\{\bar{\Omega}'(\bar{z})\}^2$$

in the first and second equations of (8·16) are then replaced by

$$(B_3/\kappa)\bar{\Omega}'(\bar{z})\;{}^0D(z,\bar{z})+B_3'z\{\bar{\Omega}'(\bar{z})\}^2$$

and

$$-(B_3/\kappa)\bar{\Omega}'(\bar{z})\;{}^0D(z,\bar{z})-B_3''z\{\bar{\Omega}'(\bar{z})\}^2$$

respectively, where $B_3'=B_3/\kappa-B_1$ and $B_3''=B_3/\kappa-B_1''$. The conditions (8·20) then reduce to

$$[\Delta'(z)]_C=0,\quad [\bar{\delta}''(\bar{z})]_C=0,$$

$$[\kappa\Delta(z)-\bar{\delta}'(\bar{z})]_C=\left[B_4\int^z\{\Omega'(z)\}^2\,dz-B_1'\int^{\bar{z}}\bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,d\bar{z}\right]_C.$$

A similar remark applies to (7·12), (7·14), (7·18), (7·24), (7·25), (8·21), (10·1) and the associated conditions for single-valuedness.

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